W*-K-SYSTEMS

Dissertation
der Mathematischen Fakultät
der Eberhard-Karls-Universität
zu Tübingen

zur Erlangung des Grades eines
Doktors der Naturwissenschaften

vorgelegt von
Wolfgang Schröder
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Vorwort

Die vorliegende Arbeit gehört in das Gebiet der nicht-kommutativen Ergodentheorie.
In der klassischen Ergodentheorie betrachtet man typischerweise eine bi-meßbare, maßerhaltende Bijektion eines Wahrscheinlichkeitsraumes (Ω, Σ, μ). Funktionalanalytisch gesehen handelt es sich dabei um einen Automorphismus auf der kommutativen W^*-Algebra $L^\infty(\Omega, \Sigma, \mu)$, der μ (als Linearform auf $L^\infty(\Omega, \Sigma, \mu)$ betrachtet) invariant läßt. In der nicht-kommutativen Ergodentheorie ersetzt man nun $L^\infty(\Omega, \Sigma, \mu)$ durch eine beliebige W^*-Algebra $\mathcal{A}$ und μ durch einen treuen normalen Zustand φ auf $\mathcal{A}$.

Wie die klassische Ergodentheorie ihre Entstehung zum großen Teil Problemen der Statistischen Mechanik verdankt, so wurde die nicht-kommutative Verallgemeinerung wesentlich durch die Theorie quantenmechanischer Vielteilchen-Systeme angeregt: Im sogenannten operator-algebraischen Formalismus beschreibt man die Observablen eines quantenmechanischen Vielteichen-Systems durch die selbstadjungierten Elemente einer W^*-Algebra $\mathcal{A}$, einen gegebenen thermodynamischen Gleichgewichtszustand durch einen treuen normalen Zustand φ auf $\mathcal{A}$ und schließlich die Dynamik durch eine stetige Einparametergruppe α von Automorphismen

auf $\mathcal{A}$, die φ invariant lassen.

Zur Untersuchung solcher Modelle $(\mathcal{A}, \varphi, \alpha)$ wurden anfangs hauptsächlich die grundlegenden Begriffe und Resultate, die Mischungsverhalten beschreiben, in die nicht-kommutative Situation übertragen. So heißt $(\mathcal{A}, \varphi, \alpha)$ etwa stark mischend, wenn

$$\lim_{|t|\to\infty} \varphi(\alpha_t(x)y) - \varphi(x)\varphi(y) = 0 \quad \text{für alle} \quad x, y \in \mathcal{A}.$$

Anschaulich gesprochen bedeutet diese Eigenschaft ein Zerfallen von Korrelationen im Laufe der Zeit.

In den letzten Jahren hat sich das Interesse an nicht-kommutativen Verallgemeinerungen auch komplexeren Teilgebieten der Ergodentheorie wie etwa Ergodensätzen, Entropie, K-Systemen und der Analyse von Dilatationen zugewandt.
Der Begriff des K-Systems, der Ende der fünfziger Jahre von A.N. Kolmogorof eingeführt wurde, beinhaltet vereinfacht gesagt ein gleichmäßiges Zerfallen von Korrelationen. Dieses Konzept gab der klassischen Ergodentheorie nachhaltige Anstösse. Zu erwähnen ist beispielsweise Sinais Beweis, daß das Billard-Gas ein K-System bildet, oder Ornsteins Konstruktion eines K-Systems, das nicht konjugiert zu einem Bernoulli-System ist.

Die bedeutende Rolle, die der Begriff des K-Systems in der klassischen Ergodentheorie und bei der Analyse von Modellen der Statistischen Mechanik spielt, legte es nahe, ein Analogon für die nicht-kommutative Theorie und ihre quantenphysikalischen Anwendungen zu formulieren. Dieser Aufgabe ist G.G. Emch zwischen 1973 und 1978 in einer Reihe von Artikeln nachgegangen (vgl. [Em1] und dortige Verweise, [Em3]). Die von Emch eingeführten "verallgemeinerten K-Flüsse" (generalized K-flows) sind definiert durch naheliegende W^*-algebraische Entsprechungen der klassischen Axiome sowie durch eine - im Kommutativen triviale - Zusatzannahme, die die Existenz einer gewissen bedingten Erwartung voraussetzt und im folgenden als "Erwartungshypothese" (expectation hypothesis) bezeichnet wird.

Bei unserer Definition von "W^*-K-Systemen" verzichten wir auf diese Zusatzannahme und gehen lediglich von den direkten Entsprechungen der drei klassischen Axiome aus. Dieser Ansatz wird dadurch gerechtfertigt, daß wir einerseits eine Vielzahl von W^*-K-Systemen konstruieren können, bei denen die Erwartungshypothese verletzt ist. Andererseits können wir auch für beliebige W^*-K-Systeme unabhängig von der Erwartungshypothese eine Analyse der Mischungseigenschaften durchführen, die erheblich über die bekannten Resultate für verallgemeinerte K-Flüsse

hinausgeht. Auch im Hinblick auf Anwendungen in der Gleichgewichtstheorie von quantenmechanischen Vielteilchen-Systemen ist ein Verzicht auf die Erwartungshypothese von Vorteil. Wird nämlich ein Quantensystem im Gleichgewicht wie oben durch ein Tripel $(\mathfrak{A}, \varphi, \alpha)$ beschrieben, so ist α typischerweise gerade die sogenannte modulare Automorphismengruppe zu $(\mathfrak{A}, \varphi)$. Falls nun $(\mathfrak{A}, \varphi, \alpha)$ ein W^*-K-System bildet, dann ist es aufgrund eines Satzes von Takesaki ausgeschlossen, daß die Erwartungshypothese gilt (vgl. 2.2.3).

W^*-K-Systeme besitzen eine gewisse Ähnlichkeit mit den vielfältig untersuchten quasilokalen Algebren (vgl. [Br1], p. 118 f), die jedoch eine wesentlich reichere Struktur besitzen. Darüberhinaus beruhen die bekannten Sätze über quasilokale Algebren in ihren topologischen Aspekten in der Regel auf der C^*-Norm, während wir bei W^*-K-Systemen mit wesentlich schwächeren Operator-Topologien arbeiten müssen.

Nachdem wir in Abschnitt 1 unsere Notation und grundlegenden Ergebnisse aus der Theorie der W^*-Algebra zusammengestellt haben, führen wir in Abschnitt 2 die zentralen Begriffe dieser Arbeit ein.

Zur Veranschaulichung und Abgrenzung verschiedener Definitionen werden im folgenden Abschnitt drei Klassen von Beispielen für W*-K-Systeme beschrieben: W*-Bernoulli-Systeme und Markov-W*-Dilatationen sind nicht-kommutative Verallgemeinerungen von Standard-Beispielen der klassischen Theorie. Mit Hilfe einer quasifreien Konstruktion über den kanonischen Anti-Vertauschungsrelationen (CAR) finden wir als dritte Klasse im Abschnitt 3.3 eine neue, wesentlich nicht-kommutative Familie von W*-K-Systemen. In dieser Klasse läßt sich ein einfaches Kriterium für die Gültigkeit der Erwartungshypothese angeben. Man kann sogar eine Vielzahl von quasifreien W*-K-Systemen finden, die - grob gesprochen - weit davon entfernt sind, die Erwartungshypothese zu erfüllen; darunter sind erste Beispiele, bei denen sogar die modulare Automorphismengruppe ein W*-K-System bildet. In einer Anwendung zeigen wir, daß eine quasifreie minimale Dilatation über den CAR unter einer natürlichen Voraussetzung zu einem verallgemeinerten K-Fluß führt. - Weitere Beispiele von W*-K-Systemen ergeben sich aus den oben genannten mit Hilfe von elementaren Operationen, die in Abschnitt 4 beschrieben werden.

Der zweite Teil der Arbeit beschäftigt sich mit einer Analyse des asymptotischen Verhaltens von W*-K-Systemen

und untersucht die Frage, inwieweit sich die bekannte Hierarchie von Mischungseigenschaften (vgl. 2.1.2) auf W*-K-Systeme erweitern läßt. Der von Emch benutzte Zugang besteht darin, im zugehörigen GNS-Hilbertraum eine Struktur zu konstruieren, die genau analog zu derjenigen eines K-Systems ist. Während sich dieser Ansatz nicht auf beliebige W*-K-Systeme übertragen läßt (vgl. 5.1), können wir im Abschnitt 5.2 mit Hilfe der Tomita-Takesaki-Theorie das gewünschte Resultat für einen modifizierten Hilbertraum beweisen.

Die Benutzung dieses Hilbertraumes ermöglicht im Abschnitt 6.1 einen neuen einfachen Beweis für die Äquivalenz von verschiedenen sehr starken Mischungseigenschaften, die in ähnlicher Form eine wichtige Rolle in der Theorie quasilokaler Algebren spielen. Als Korollar erhalten wir für W*-K-Systeme eine Art gleichmäßiges Mischen, ähnlich dem K-Mischen in der klassischen Theorie. Die naheliegende nicht-kommutative Verallgemeinerung von K-Mischen trifft jedoch auf W*-K-Systeme im allgemeinen nicht zu (vgl. 6.2). Nichtsdestoweniger können wir im letzten Abschnitt zeigen, daß jedes W*-K-System, unabhängig von der Erwartungshypothese, einer Eigenschaft genügt, die Mischen von beliebigem Grad entspricht.

Zum Entstehen dieser Arbeit hat die kollegiale Atmosphäre in den beiden Arbeitsbereichen "Funktionalanalysis" und "Mathematische Methoden der Physik" vieles beigetragen. Vor allem danke ich Herrn Prof. R. Nagel, dem Betreuer dieser Arbeit, und Herrn Prof. M. Wolff für ihre Unterstützung, Herrn Dr. B. Kümmerer für vielfältige Anregungen und Herrn J. Vázquez für kritische Diskussionen. Schließlich gilt mein Dank auch Herrn Prof. G.G. Emch (Rochester) für sein Interesse an meiner Arbeit.

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1 Preliminaries

The terminology of the present work generally conforms to the books of S.Sakai [Sa] and M.Takesaki [Ta2]. The following notation and basic results will be used throughout the subsequent chapters.

Let $\mathcal{A}$ be a W*-algebra. We shall write

$\mathcal{A}^{sa} := \{x \in \mathcal{A} : x^* = x\}$ for its self-adjoint part,

$\mathcal{A}^{+} := \{x^*x : x \in \mathcal{A}\}$ for its positive cone and

$\mathcal{A}^{1} := \{x \in \mathcal{A} : \|x\| \leq 1\}$ for its unit ball.

The identity of $\mathcal{A}$ is denoted by $\mathbb{1}_{\mathcal{A}}$ or briefly by $\mathbb{1}$.

The W*-algebra of all bounded linear operators on a Hilbert space $\mathcal{H}$ is denoted by $\mathcal{B}(\mathcal{H})$. Given $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$ the commutant of $\mathcal{M}$ in $\mathcal{B}(\mathcal{H})$ is defined as $\mathcal{M}' := \{y \in \mathcal{B}(\mathcal{H}) : xy = yx \text{ for all } x \in \mathcal{M}\}$.

For the predual of $\mathcal{A}$ we use the symbol $\mathcal{A}_*$.

A normal state on $\mathcal{A}$ is an element ψ of $\mathcal{A}_*$ satisfying $\psi(x) \geq 0$ for $x \in \mathcal{A}^{+}$ and $\psi(\mathbb{1}) = 1$.

It is faithful if $\psi(x) = 0$ for $x \in \mathcal{A}^{+}$ implies $x = 0$.

The weak* topology $\sigma(\mathcal{A}, \mathcal{A}_*)$ on $\mathcal{A}$ is the weakest topology rendering each $\psi \in \mathcal{A}_*$ continuous.

The strong topology $s(\mathcal{A}, \mathcal{A}_*)$ is the one generated

by the seminorms $p_\psi(x) := \psi(x^*x)^{1/2}$ where ψ runs over the normal states on $\mathcal{A}$. For a faithful normal state φ on $\mathcal{A}$, p_φ is a norm on $\mathcal{A}$. Under the corresponding metric $\mathcal{A}^1$ is complete, and the topology generated on $\mathcal{A}^1$ coincides with the restriction of $s(\mathcal{A}, \mathcal{A}_*)$ (cf. [Ta2], p.148).
The strong* topology $s^*(\mathcal{A}, \mathcal{A}_*)$ is the one generated by the family of seminorms $x \longmapsto (p_\psi(x) + p_\psi(x^*))^{1/2}$ indexed by the normal states ψ on $\mathcal{A}$.
On $\mathcal{B}(\mathcal{H})$ the strong operator topology, called stop for short, is given by the seminorms $x \longmapsto \|x\eta\|$ with η running through $\mathcal{H}$.

By a W*-subalgebra of $\mathcal{A}$ we mean a weak* closed *-subalgebra of $\mathcal{A}$ containing the identity of $\mathcal{A}$. If $(\mathcal{R}_\nu)_{\nu \in I}$ is a family of subsets of $\mathcal{A}$, $\bigvee\{\mathcal{R}_\nu : \nu \in I\}$ denotes the W*-subalgebra of $\mathcal{A}$ generated by $(\mathcal{R}_\nu)_{\nu \in I}$, i.e. the intersection of all W*-subalgebras of $\mathcal{A}$ containing each $\mathcal{R}_\nu$.

Given a faithful normal state φ on $\mathcal{A}$ we call the pair $(\mathcal{A}, \varphi)$ a W*-algebra with faithful normal state. An automorphism of $(\mathcal{A}, \varphi)$ is a *-automorphism of $\mathcal{A}$ which leaves φ invariant. We denote by $\mathrm{Aut}(\mathcal{A}, \varphi)$ the group of automorphisms of $(\mathcal{A}, \varphi)$.

Let σ be the modular group associated with $(\mathfrak{A}, \varphi)$ (cf. [Br1], p.96; [St], p.292). A W^*-subalgebra $\mathfrak{M}$ of $\mathfrak{A}$ is globally invariant under σ if and only if there is a projection of norm 1 from $\mathfrak{A}$ onto $\mathfrak{M}$ leaving φ invariant [Ta1]. Such a projection is unique and will be called the conditional expectation of $(\mathfrak{A}, \varphi)$ onto $\mathfrak{M}$.

Let $\mathbb{T}$ be the group $\mathbb{Z}$ or $\mathbb{R}$ with the standard topology and order, and define

$\mathbb{T}^- := \{ t \in \mathbb{T} : t \leq 0 \}$,

$\mathbb{T}^+ := \{ t \in \mathbb{T} : t \geq 0 \}$.

By a representation α of $\mathbb{T}$ in Aut $(\mathfrak{A}, \varphi)$ we mean a continuous group homomorphism of $\mathbb{T}$ into Aut $(\mathfrak{A}, \varphi)$ equipped with the topology of pointwise weak* convergence; the image of $t \in \mathbb{T}$ under α is always denoted by α_t, no matter whether $\mathbb{T} = \mathbb{Z}$ or $\mathbb{T} = \mathbb{R}$. Correspondingly we define a unitary representation of $\mathbb{T}$ on a Hilbert space $\mathfrak{H}$ and also a representation of the semigroup $\mathbb{T}^+$ as completely positive operators on $\mathfrak{A}$ or as contractions on $\mathfrak{H}$.

Given a representation α of $\mathbb{T}$ in Aut $(\mathfrak{A}, \varphi)$ we call the triple $(\mathfrak{A}, \varphi, \alpha)$ a W^*-system. If we want to emphasize that $\mathbb{T} = \mathbb{Z}$ (resp. $\mathbb{T} = \mathbb{R}$) we

talk of a discrete (resp. continuous) W^*-system. The set of fixed points of α in $\mathcal{A}$ is denoted by Fix $(\mathcal{A}, \alpha)$.

Let $(\mathcal{A}, \varphi)$ be a W^*-algebra with faithful normal state. With $(\mathcal{A}, \varphi)$ the GNS-construction canonically associates a Hilbert space $\mathcal{H}$ with scalar product $\langle . \mid . \rangle$ and a unit vector $\xi \in \mathcal{H}$ such that the following conditions hold:
The W^*-algebra $\mathcal{A}$ can be regarded as a von Neumann algebra on $\mathcal{H}$, the vector $\xi \in \mathcal{H}$ is cyclic and separating for $\mathcal{A}$, and
$\varphi(x) = \langle x\xi \mid \xi \rangle$ for all $x \in \mathcal{A}$ (cf. [Br1], p.56). We call $(\mathcal{H}, \xi)$ the GNS-pair associated with $(\mathcal{A}, \varphi)$.

If $(\mathcal{A}, \varphi, \alpha)$ is a W^*-system and $(\mathcal{H}, \xi)$ the GNS-pair associated with $(\mathcal{A}, \varphi)$ then there is a unique unitary representation U of $\mathbb{T}$ on $\mathcal{H}$ such that
$\alpha_t(x) = U_t x U_t^*$ and
$U_t \xi = \xi$ for all $x \in \mathcal{A}$, $t \in \mathbb{T}$.
We call $(\mathcal{H}, \xi, U)$ the GNS-triple associated with $(\mathcal{A}, \varphi, \alpha)$.

2 Introduction

2.1 K-systems in classical ergodic theory

An intensive development of ergodic theory was initiated in 1958 by an article of A.N.Kolmogorov [Ko] introducing two closely related concepts - entropy and so called quasi-regular systems, later named K-systems.

In his address to the International Congress of Mathematicians, Stockholm 1962, Ya.G.Sinai [Si1] emphasized that the notion of a K-system opened up a new way for studying measure-preserving transformations : Probabilistic ideas which had originally been elaborated for stationary stochastic processes were now being transferred to ergodic theory.

2.1.1 For recalling these probabilistic ideas let us consider a stochastic process $(\Omega, \Sigma, \mu, (X_t)_{t \in \mathbb{T}})$ with parameter set $\mathbb{T}$. (When we talk of stochastic processes we use the standard terminology as it is fixed e.g. in [Gi].) For simplicity we assume that Σ is generated by $(X_t)_{t \in \mathbb{T}}$.
Relevant aspects of the stochastic process may be studied in terms of the family $(\Sigma_{t]})_{t \in \mathbb{T}}$ of σ-subalgebras of Σ where each $\Sigma_{t]}$ is generated by $(X_s)_{s \leq t}$. In particular, the asymptotic behaviour

of the stochastic process can be described by the following concept :

The stochastic process is said to satisfy the <u>0-1-law</u> if $\bigcap \{ \Sigma_{t]} : t \in \mathbb{T} \} = \{ \emptyset , \Omega \}$; alternatively such a stochastic process is called regular (cf. [Gi] , p.143; [Si1]) .

(Here and in the following all statements about inclusion and equality of σ-algebras in a probability space are to be understood in the sense of measure algebras.)

If in addition $(\Omega , \Sigma , \mu , (X_t)_{t \in \mathbb{T}})$ is stationary (cf. [Gi] , p.132) then there is a representation T of $\mathbb{T}$ as bi-measurable, μ-preserving bijections on Ω such that

$X_s \circ T_t = X_{s-t}$ for all $s,t \in \mathbb{T}$.

In this situation we can describe $(\Sigma_{t]})_{t \in \mathbb{T}}$ simply by $\Sigma_{0]}$ and T since

$\Sigma_{t]} = T_t \ \Sigma_{0]}$ for all $t \in \mathbb{T}$.

Now the 0-1-law reads

$\bigcap \{ T_t \ \Sigma_{0]} : t \leq 0 \} = \{ \emptyset , \Omega \}$,

while our assumption that Σ is generated by $(X_t)_{t \in \mathbb{T}}$ takes the form

$\bigvee \{ T_t \ \Sigma_{0]} : t \geq 0 \} = \Sigma$.

(Whenever a family $(\Sigma_\nu)_{\nu \in I}$ of σ-subalgebras of a certain σ-algebra is given, we denote by

$\bigvee\{\Sigma_\nu : \nu \in I\}$ the σ-subalgebra generated by the Σ_ν, i.e. the intersection of all σ-subalgebras containing each Σ_ν .)

2.1.2 It was Kolmogorov who suggested to employ analogous conditions in the study of an abstract measure theoretical dynamical system (Ω, Σ, μ, T) consisting of a probability space (Ω, Σ, μ) and a representation T of $\mathbb{T}$ as bi-measurable, μ-preserving bijections on Ω. He considered (Ω, Σ, μ, T) together with a σ-subalgebra Σ_K of Σ such that the following conditions hold :

(CK1) $\Sigma_K \subseteq T_t \Sigma_K$ for all $t \geq 0$,

(CK2) $\bigvee\{T_t \Sigma_K : t \geq 0\} = \Sigma$,

(CK3) $\bigcap\{T_t \Sigma_K : t \leq 0\} = \{\emptyset, \Omega\}$.

Later such a structure $(\Omega, \Sigma, \mu, T ; \Sigma_K)$ became known as a <u>K-system</u> (cf. [Co], p.280) .

The new concept turned out to be rather useful for ergodic theory, and beyond the realm of stochastic processes numerous examples of K-systems were found, in particular relevant models of statistical mechanics were shown to be K-systems (cf. [Co], p.281) .

Instead of dwelling on the close relations with the

theory of entropy we shall rather concentrate on the aspect how the notion of a K-system fits into the well-known hierarchy of mixing properties. In fact, the following chain of implications holds:

Bernoulli-system
$\Longrightarrow$ K-system
$\Longleftrightarrow$ K-mixing $\Longrightarrow$ homogeneous
$\Longrightarrow$ mixing of arbitrary degree Lebesgue spectrum
$\Longrightarrow$ strong mixing $\Longleftarrow$
$\Longrightarrow$ weak mixing
$\Longrightarrow$ ergodic

(cf. [Co] , for a particularly concise account of the discrete case cf. [Wa] , for a formulation in the spirit of functional analysis cf. [De]).

2.2 K-systems in non-commutative ergodic theory

The usefulness of K-systems in classical ergodic theory suggested to introduce an appropriate analogue into non-commutative ergodic theory and the analysis of models in quantum physics. From 1973 onwards G.G. Emch elaborated this idea in a series of articles (cf. [Em1] and references therein). Another motivation for studying non-commutative K-systems may be drawn from work of L.Accardi [Ac1] and L.Accardi, A.Frigerio and J.T.Lewis [Ac2] who began

to investigate non-commutative versions of stochastic processes.

2.2.1 No matter which of the various existing definitions of a non-commutative stochastic process (cf. e.g. [Ac1] , [Ac2]) one may prefer, for any W^*-algebraic extension of the theory of stochastic processes it should be relevant to consider the following analogue of the situation in 2.1.1 : Instead of a probability space (Ω, Σ, μ) take a pair $(\mathcal{A}, \varphi)$, a W^*-algebra with faithful normal state. In place of the family $(\Sigma_{t])_{t \in \mathbb{T}}}$ consider a family $(\mathcal{A}_{t])_{t \in \mathbb{T}}}$ of W^*-subalgebras of $\mathcal{A}$ such that $\mathcal{A}_{s]} \subseteq \mathcal{A}_{t]}$ for $s,t \in \mathbb{T}$, $s \leq t$, and $\vee \{\mathcal{A}_{t]} : t \in \mathbb{T}\} = \mathcal{A}$.
Of particular interest is the case when in addition a W^*-algebraic version of the 0-1-law holds :
$\cap \{\mathcal{A}_{t]} : t \in \mathbb{T}\} = \mathbb{C}\mathbb{I}$.
An analysis of such a situation in a general frame of non-commutative stochastic processes is certainly not the main intention of this work. Nevertheless, as a by-product our analysis of non-commutative K-systems will also provide results about this general setting as we shall see in chapters 5 and 6.

2.2.2 Our main interest is devoted to a more specific situation corresponding to a stationary stochastic process or a measure theoretical dynamical system. So together with a pair $(\mathcal{A} , \varphi)$, a W^*-algebra with faithful normal state, we shall usually consider a representation α of $\mathbb{T}$ in $Aut (\mathcal{A} , \varphi)$, i.e. the basic object will be a W^*-system $(\mathcal{A} , \varphi , \alpha)$.

For developing a W^*-algebraic analogue of a K-system $(\Omega , \Sigma , \mu , T ; \Sigma_K)$ let us put

$\mathcal{M} := L^\infty(\Omega , \Sigma , \mu)$, $\mu (f) := \int_\Omega f \, d\mu$

$\tau_t (f) := f \circ T_t^{-1}$ for $f \in \mathcal{M}$, $t \in \mathbb{T}$, and finally

$\mathcal{M}_K := L^\infty(\Omega , \Sigma_K , \mu)$.

If we now reformulate the properties (CK1) - (CK3) of 2.1.2 for the W^*-system $(\mathcal{M} , \mu , \tau)$ together with the W^*-subalgebra $\mathcal{M}_K$ of $\mathcal{M}$ we arrive at the following notions which will form the fundament of the present work :

Definitions

Let $(\mathcal{A} , \varphi , \alpha)$ be a W^*-system and $\mathcal{A}_K$ a W^*-subalgebra of $\mathcal{A}$ such that the following conditions hold :

(K1) $\mathcal{A}_K \subseteq \alpha_t \mathcal{A}_K$ for all $t \geq 0$,

(K2) $\vee \{ \alpha_t \mathcal{A}_K : t \geq 0 \} = \mathcal{A}$.

Then we call $(\mathcal{A} , \varphi , \alpha ; \mathcal{A}_K)$ an expanding W^*-system and $\mathcal{A}_K$ an expanding W^*-subalgebra.

Furthermore $(\mathfrak{A}, \varphi, \alpha; \mathfrak{A}_K)$ is said to satisfy the 0-1-property if in addition

(K3) $\bigcap\{\alpha_t \mathfrak{A}_K : t \leq 0\} = \mathbb{C}1$.

In that case $(\mathfrak{A}, \varphi, \alpha; \mathfrak{A}_K)$ is called a W*-K-system.

Remarks

(1) When we constructed the W*-system $(\mathfrak{M}, \mu, \tau)$ above from the measure theoretical dynamical system $(\Omega, \Sigma, \mu; T)$ we defined $\tau_t(f) := f \circ T_t^{-1}$ for $f \in \mathfrak{M}$, $t \in \mathbb{T}$ while it is more customary (cf. [De], p.II/4) to study the representation τ^{-1} defined by $\tau_t^{-1}(f) := f \circ T_t$ for $f \in \mathfrak{M}$, $t \in \mathbb{T}$. With respect to τ^{-1} the W*-subalgebra $\mathfrak{M}_K$ of $\mathfrak{M}$ is of course "contracting", i.e.

$\mathfrak{M}_K \supseteq \tau_t^{-1}\mathfrak{M}_K$ for all $t \geq 0$,

and correspondingly we have

$\bigvee\{\tau_t^{-1}\mathfrak{M}_K : t \leq 0\} = \mathfrak{M}$,

$\bigcap\{\tau_t^{-1}\mathfrak{M}_K : t \geq 0\} = \mathbb{C}1$.

Clearly, the difference between these properties and the defining conditions of a W*-K-system does not matter at all as long as we study just the quadrupels $(\mathfrak{M}, \mu, \tau; \mathfrak{M}_K)$ or $(\mathfrak{M}, \mu, \tau^{-1}; \mathfrak{M}_K)$. However, if a W*-system is given as part of a larger structure an expanding W*-subalgebra may be rather more natural than a contracting one. This is in

particular the case when one deals with so called Markov-W^*-dilations (cf. 3.2) which provide a wealth of W^*-K-systems. We had mostly applications to Markov-W^*-dilations in mind (cf. 3.2) when we chose to define a W^*-K-system by means of an expanding instead of a contracting W^*-subalgebra.
This choice is also consistent with the definition of G.G.Emch [Em1].
(2) While the conditions (K1) - (K3) do not explicitly refer to an invariant faithful normal state, in general we cannot do without such a state if we want to conclude from (K1) - (K3) only the weakest of the usual mixing properties (cf. 2.1.2) , namely ergodicity.
This is demonstrated by the following example :
Let $L^2(\mathbb{T})$ be the Hilbert space of (equivalence classes of) functions on $\mathbb{T}$ which are square integrable with respect to Haar measure on $\mathbb{T}$. Define $L^2(\mathbb{T}^-)$ as the closed subspace of $L^2(\mathbb{T})$ consisting of functions with essential support in $\mathbb{T}^-$.
Let T be the representation of $\mathbb{T}$ on $L^2(\mathbb{T})$ with
$(T_t f)(s) := f(s-t)$ for all $s,t \in \mathbb{T}$, $f \in L^2(\mathbb{T})$.

Now consider $\mathfrak{A} := \mathfrak{B}(L^2(\mathbb{T}))$ with the representation τ of $\mathbb{T}$ in $\mathrm{Aut}(\mathfrak{A})$ defined by

$\tau_t(x) := T_t x T_t^*$ for all $x \in \mathfrak{N}$.

Denote by $p \in \mathfrak{N}$ the projection onto $L^2(\mathbb{T}^-)$ and put $\mathfrak{N}_{0]} := p\,\mathfrak{N}\,p + \mathbb{C}(\mathbb{1} - p)$.

(We have included the direct summand $\mathbb{C}(\mathbb{1} - p)$ in the definition of $\mathfrak{N}_{0]}$ in order to make sure that the identity of $\mathfrak{N}$ is in $\mathfrak{N}_{0]}$.)

While there is no τ-invariant normal state on $\mathfrak{N}$ one easily checks that

$\mathfrak{N}_{0]} \subseteq \tau_t\,\mathfrak{N}_{0]}$ for all $t \geq 0$,

$\bigvee \{\tau_t\,\mathfrak{N}_{0]} : t \geq 0\} = \mathfrak{N}$ and

$\bigcap \{\tau_t\,\mathfrak{N}_{0]} : t \leq 0\} = \mathbb{C}\,\mathbb{1}$.

Indeed, $p \leq \tau_t(p)$ for all $t \geq 0$ and

$\text{stop-}\lim_{t\to\infty} \tau_t(p) = \mathbb{1}$, $\text{stop-}\lim_{t\to-\infty} \tau_t(p) = 0$.

However, the action of τ on $\mathfrak{N}$ is obviously far from being ergodic. While

$\bigcup \{\tau_t\,\mathfrak{N}_{0]} : t \geq 0\} \cap \mathrm{Fix}(\mathfrak{N}, \tau) = \mathbb{C}\,\mathbb{1}$

holds we obtain $\mathrm{Fix}(\mathfrak{N}, \tau) = \{T\}' = \{T_t : t \in \mathbb{T}\}'$ in the weak* closure $\mathfrak{N}$ of $\bigcup \{\tau_t\,\mathfrak{N}_{0]} : t \geq 0\}$.

Via Fourier transform $\mathrm{Fix}(\mathfrak{N}, \tau)$ is isomorphic to $L^\infty(\hat{\mathbb{T}})$, the W*-algebra of all functions on the dual group $\hat{\mathbb{T}}$ of $\mathbb{T}$ which are essentially bounded with respect to Haar measure on $\mathbb{T}$.

Note that there cannot exist a normal projection of norm 1 from $\mathfrak{N}$ onto $\mathrm{Fix}(\mathfrak{N}, \tau)$ because

Fix ($\mathcal{M}$, τ) contains no minimal projections [To].

Later in our analysis of a W^*-K-system ($\mathcal{A}$, φ , α ; $\mathcal{A}_K$) we shall use just the existence of the conditional expectation of ($\mathcal{A}$, φ) onto Fix ($\mathcal{A}$, α) to conclude in a straightforward way that in this case Fix ($\mathcal{A}$, α) = $\mathbb{C}$ $\mathbb{1}$ (cf. 6.3).

2.2.3 Our definition of a W^*-K-system comprises only just those requirements which one may regard as a minimum for a W^*-algebraic generalization of a K-system. However, do these conditions suffice for a theory which is in any way comparable to the wealth of the classical theory ? In particular, do W^*-K-systems fit into the hierarchy of mixing properties in non-commutative ergodic theory analogous to the chain of implications in 2.1.2 ?
As the essential step for answering these questions we want to show in chapter 5 that any W^*-K-system ($\mathcal{A}$, φ , α ; $\mathcal{A}_K$) satisfies a property corresponding to homogeneous Lebesgue spectrum in classical ergodic theory.
For proving that a K-system (Ω , Σ , μ , T ; Σ_K) has homogeneous Lebesgue spectrum the key idea is to see how the closed subspace $L^2(\Omega, \Sigma_K, \mu)$ of

$L^2(\Omega, \Sigma, \mu)$ is shifted by the unitary representation induced by T (cf. [Co], p.338; [Si2]).
For a non-commutative generalization one might consider the GNS-triple $(\mathfrak{X}, \xi, U)$ corresponding to $(\mathfrak{A}, \varphi, \alpha)$, replace $L^2(\Omega, \Sigma_K, \mu)$ by the closure $\mathfrak{X}_K$ of $\mathfrak{A}_K\xi$ in $\mathfrak{X}$ and then try to verify the following analogues of the classical properties :

$\mathfrak{X}_K \subseteq U_t \mathfrak{X}_K$ for all $t \geq 0$,

$\vee \{U_t \mathfrak{X}_K : t \geq 0\} = \mathfrak{X}$,

$\cap \{U_t \mathfrak{X}_K : t \leq 0\} = \mathbb{C}\xi$ (cf. [Em1]).

(Whenever a family $(\mathfrak{L}_\nu)_{\nu \in I}$ of subsets of a Hilbert space is given we denote by $\vee\{\mathfrak{L}_\nu : \nu \in I\}$ the smallest closed subspace containing each $\mathfrak{L}_\nu$.)
However, while the first two properties are easily verified the proof of the last property presents a problem.
Actually, we shall demonstrate in 5.1 that there are W^*-K-systems for which $\mathfrak{X}_K = \mathfrak{X}$.
Yet G.G.Emch [Em1] was able to establish this last property under the additional requirement that the conditional expectation of $(\mathfrak{A}, \varphi)$ onto $\mathfrak{A}_K$ exists.

Definition

We say that an expanding W^*-system $(\mathfrak{A}, \varphi, \alpha; \mathfrak{A}_K)$ satisfies the expectation hypothesis if the conditional

expectation of $(\mathcal{A}, \varphi)$ onto $\mathcal{A}_K$ exists.

Remark

For $(\mathcal{A}, \varphi, \alpha; \mathcal{A}_K)$ the expectation hypothesis holds if and only if the modular group associated with $(\mathcal{A}, \varphi)$ leaves $\mathcal{A}_K$ globally invariant [Ta1]. Obviously, the expectation hypothesis is satisfied if $\mathcal{A}$ is commutative.

In his study of W^*-K-systems satisfying the expectation hypothesis Emch [Em1] concentrated on the case $\mathbb{T} = \mathbb{R}$ and called such systems generalized K-flows. Since we also want to cover the case $\mathbb{T} = \mathbb{Z}$ we prefer the following terminology :

Definition

A W^*-K-system satisfying the expectation hypothesis will be called a (discrete or continuous) generalized K-system.

Various examples of continuous generalized K-systems have been worked out by Emch et al. [Em1] , [Em3] . As we shall see in section 3.2 a large class of discrete generalized K-systems can be found by a non-commutative dilation scheme developed in [KÜ1] . Another method for producing generalized K-systems will be demonstrated in section 3.3 - namely a quasi-

free construction over the canonical anti-commutation relations (CAR) .

A close relation between generalized K-systems and non-equilibrium quantum statistical mechanics is outlined in the introduction of [Em1] . In [Em2] these ideas are explicitly worked out for the diffusion of a quantum particle in a harmonic well.

A somewhat different view of the role generalized K-systems may play in quantum physics is proposed in [Kü4] , namely that they arise from stationary quantum stochastic processes in a manner analogous to that described in 2.1 . Along such a line we have in some detail discussed various generalized K-systems related to the relaxation of a spin-$\frac{1}{2}$ governed by the Bloch equations [Kü3] , [Kü4] .

In spite of the fact that the expectation hypothesis is satisfied by a wide range of examples and is certainly adequate in applications of W^*-K-systems to non-equilibrium quantum statistical mechanics there are various reasons for studying W^*-K-systems in a way which does not a priori rely on the expectation hypothesis.

One reason is simply our observations that there exists a variety of W^*-K-systems which actually violate the expectation hypothesis. In section 3.3 a class of such

W^*-K-systems will be constructed explicitly in a quasifree manner over the CAR. This construction especially yields W^*-K-systems $(\mathcal{A}, \varphi, \sigma; \mathcal{A}_K)$ where σ is the modular group associated with $(\mathcal{A}, \varphi)$ (cf. 3.3.6). By a theorem of Takesaki [Ta1] one sees immediately that in such a case the conditional expectation of $(\mathcal{A}, \varphi)$ onto $\mathcal{A}_K$ cannot exist. A method for generating further examples of W^*-K-systems violating the expectation hypothesis is provided by the direct product described in section section 4.1 .

Another argument for developing an ergodic theory of W^*-K-systems independently of the expectation hypothesis results from our intention to widen the range of possible applications in quantum physics. This range should in our mind include not only the non-equilibrium theory as mentioned above, but also the equilibrium theory of quantum statistical mechanics. For motivation we recall that a number of classical models in equilibrium statistical mechanics are in fact K-systems, e.g. the ideal gas (cf. [Co], p.281). However, in the operator-algebraic formalism quantum systems in equilibrium are typically described by a W^*-system $(\mathcal{A}, \varphi, \sigma)$ where σ denotes the modular group associated with $(\mathcal{A}, \varphi)$ (cf. [Br2]). If $\mathcal{A}_K$ is now an expanding subalgebra for

$(\alpha, \varphi, \sigma)$ then the conditional expectation of (α, φ) onto α_K cannot exist as we have just seen.

So in view of applications as well as for purely mathematical reasons it seems interesting to strive for an ergodic theory of W^*-K-systems which does not depend on the expectation hypothesis as a basic assumption.

3 Examples of W*-K-systems

3.1 W*-Bernoulli-systems

In classical ergodic theory the certainly most lucid examples of K-systems are Bernoulli-shifts. What follows may be regarded as a natural non-commutative analogue of a Bernoulli-shift.

First recall the construction of the appropriate infinfite tensor product (cf. [Sa], p.202):

Given a family $\left((\mathcal{A}_\nu, \varphi_\nu)\right)_{\nu \in I}$ of W*-algebras with faithful normal state let $\otimes\{ \mathcal{A}_\nu : \nu \in I \}$ be the infinite C*-tensor product of the algebras $(\mathcal{A}_\nu)_{\nu \in I}$ and let $\otimes\{ \varphi_\nu : \nu \in I \}$ be the infinite product state of $(\varphi_\nu)_{\nu \in I}$ on $\otimes\{ \mathcal{A}_\nu : \nu \in I \}$ (cf. [Sa], p.75). Denote by $\hat{\mathcal{A}}$ the W*-algebra generated by the GNS-representation of $\otimes\{ \mathcal{A}_\nu : \nu \in I \}$ with respect to $\otimes\{ \varphi_\nu : \nu \in I \}$ and denote by $\hat{\varphi}$ the faithful normal state on $\hat{\mathcal{A}}$ which extends $\otimes\{ \varphi_\nu : \nu \in I \}$.

Definition

We call $(\hat{\mathcal{A}}, \hat{\varphi})$ the infinite tensor product of $\left((\mathcal{A}_\nu , \varphi_\nu)\right)_{\nu \in I}$.

For constructing a non-commutative Bernoulli-shift we now assume that $I = \mathbb{Z}$ and all the $(\mathcal{A}_k , \varphi_k)$, $k \in \mathbb{Z}$, are equal to a given $(\mathcal{A} , \varphi)$.

Let $\hat{\alpha}$ be the representation of $\mathbb{Z}$ in Aut $(\hat{\mathcal{A}} , \hat{\varphi})$

which is defined by the right shift on $\otimes\{ \alpha_k : k \in \mathbb{Z}\}$.

Definition

We call $(\hat{\alpha}, \hat{\varphi}, \hat{\alpha})$ the W*-Bernoulli-system over (α, φ).

In complete analogy to the classical theory (cf.2.1.2 or [Wa],p.107) one verifies that a W*-Bernoulli-system is a W*-K-system in the canonical way:

Proposition

Together with the expanding W*-subalgebra $\hat{\alpha}_{[0]}$ of $\hat{\alpha}$ generated by $\otimes\{ \alpha_k : k \leq 0\}$ the W*-Bernoulli-system $(\hat{\alpha}, \hat{\varphi}, \hat{\alpha})$ over (α, φ) forms a generalized K-system.

Proof:

While it is obvious that $(\hat{\alpha}, \hat{\varphi}, \hat{\alpha}; \hat{\alpha}_{[0]})$ is an expanding W*-system,for verifying the 0-1-property we translate the classical argument into the language of W*-algebras:

For $k \in \mathbb{Z}$ let $\hat{\alpha}_{[k}$ be the W*-subalgebra of $\hat{\alpha}$ generated by $\otimes\{ \alpha_n : n \geq k \}$. Given a projection $p \in \bigcap\{ \hat{\alpha}_k \hat{\alpha}_{[0]} : k \leq 0\}$ we have $\hat{\varphi}(px) = \hat{\varphi}(p) \cdot \hat{\varphi}(x)$ for all $x \in \hat{\alpha}_{[k}$, $k \in \mathbb{Z}$. Since $\bigcup\{ \hat{\alpha}_{[k} : k \in \mathbb{Z}\}$ is weak* dense in $\hat{\alpha}$ we also have $\hat{\varphi}(p) = \hat{\varphi}(p^2) = \hat{\varphi}(p)^2$. Hence $\hat{\varphi}(p) = 0$ or $\hat{\varphi}(p) = 1$ and so $p = 0$ or $p = \mathbb{1}$.

Applying a theorem of Takesaki [Ta1] (cf.Preliminaries) one checks that $(\hat{\alpha}, \hat{\varphi}, \hat{\alpha}; \hat{\alpha}_{[0]})$ satisfies the expectation hypothesis. q.e.d.

Recall that $\hat{\alpha}$ is a factor if and only if α is a factor (cf.[Sa],p.202). In the case when α is the algebra of all 2x2-matrices the type of $\hat{\alpha}$ is well known:Without loss of generality we may assume that φ is given by a density matrix of the form $\begin{pmatrix} \eta & 0 \\ 0 & 1-\eta \end{pmatrix}$ with $0 < \eta \leq \frac{1}{2}$. If $\eta = \frac{1}{2}$ then $\hat{\alpha}$ is the hyperfinite II_1-factor. For $0 < \eta < \frac{1}{2}$ one obtains the hyperfinite III_λ -factor with $\lambda := \eta \cdot (1-\eta)^{-1}$.

3.2 Markov-W*-dilations

A well studied class of K-systems which includes the Bernoulli-shifts is formed by the mixing Markov chains. The Kolmogorov-Daniell construction of a stationary Markov process from a given stochastic matrix and a given invariant distribution may in terms of functional analysis be viewed as a so called Markov dilation [Ke]. In the framework of W*-algebras this means the following: Assume we are given (α, φ) , a W*-algebra with faithful normal state,and a representation γ of $\mathbb{T}^+$ as completely positive,identity-preserving operators on α such that $\varphi \circ \gamma_t = \varphi$ holds for all $t \in \mathbb{T}^+$.

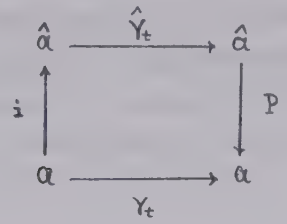
Definition

The triple $(\alpha, \varphi, \gamma)$ will be called a (discrete or continuous) dissipative W*-system.

Let $(\hat{\alpha} , \hat{\varphi} , \hat{\gamma})$ be a W^*-system and i an injection of α into $\hat{\alpha}$ such that $\hat{\varphi} \circ i = \varphi$ holds. Denote by P the uniquely determined operator from $\hat{\alpha}$ into α which satisfies $\hat{\varphi}(i(x)y) = \varphi(xP(y))$ for $x \in \alpha$, $y \in \hat{\alpha}$. Note that $i \circ P$ is the conditional expectation of $(\hat{\alpha} , \hat{\varphi})$ onto $i(\alpha)$.

Definition

The quadrupel $(\hat{\alpha}, \hat{\varphi} , \hat{\gamma} ; i)$ is called a W^*-dilation of $(\alpha , \varphi , \gamma)$ if the diagram



commutes for all $t \in \mathbb{T}^+$.

It is said to be minimal if in addition

$\bigvee \{ \hat{\gamma}_t \circ i(\alpha) : t \in \mathbb{T} \} = \hat{\alpha}$.

Put $\hat{\alpha}_{\ominus} := \bigvee \{ \hat{\gamma}_s \circ i(\alpha) : s \leq 0 \}$

and $\hat{\alpha}_{\oplus} := \bigvee \{ \hat{\gamma}_s \circ i(\alpha) : s \geq 0 \}$,

and let $P_{\ominus}$ (resp. P_o) be the conditional expectation of $(\hat{\alpha} , \hat{\varphi})$ onto $\hat{\alpha}_{\ominus}$ (resp. $i(\alpha)$) .

If $(\hat{\alpha} , \hat{\varphi} , \hat{\gamma} ; i)$ is a minimal W^*-dilation then $(\hat{\alpha} , \hat{\varphi} , \hat{\gamma} ; \hat{\alpha}_{\ominus})$ obviously forms an expanding W^*-system satisfying the expectation hypothesis.

However,if one looks for nice conditions implying the 0-1-property it is useful to impose an extra condition on the W^*-dilation.

Definition

A W^*-dilation $(\hat{\alpha} , \hat{\varphi} , \hat{\gamma} ; i)$ is said to be a Markov-W^*-dilation if $P_{\vartheta}(x) = P_0(x)$ for all $x \in \hat{\alpha}_{[\vartheta}$.

Example

The W^*-Bernoulli-system over (α , φ) together with the canonical injection of α into the zero-component of the infinite tensor product forms a minimal Markov-W^*-dilation of the discrete dissipative W^*-system $(\alpha , \varphi , \gamma)$ with $\gamma_t(x) := \varphi(x)\mathbb{1}$ for $t=1, x \in \alpha$.

Given a dissipative W^*-system $(\alpha , \varphi , \gamma)$ we look for conditions guaranteeing that a minimal Markov-W^*-dilation $(\hat{\alpha} , \hat{\varphi} , \hat{\gamma} ; i)$ exists and that $(\hat{\alpha} , \hat{\varphi} , \hat{\gamma} ; \hat{\alpha}_{\vartheta})$ forms a generalized K-system.

For special dissipative W^*-systems which are built in a quasifree manner over a representation of the canonical commutation relations (CCR) or the canonical anti-commutation relations (CAR) Markov-W^*-dilations can be produced by a quasifree construction scheme described in [Em3] and [Ev] .

In the CCR case Emch,Albeverio and Eckmann [Em3] have studied a condition on the asymptotic behaviour of $(\alpha, \varphi, \gamma)$ which implies that the quasifree minimal Markov-W*-dilation gives rise to generalized K-system. A corresponding result for the CAR case seems not to have been known. We shall deal with this problem in section 3.3.

A general theory of Markov-W*-dilations for discrete dissipative W*-systems was initiated by B.Kümmerer [Kü1] . He constructs Markov-W*-dilations for large classes of discrete dissipative W*-systems.Concerning generalized K-systems he proves the following:

Given a discrete dissipative W*-system $(\alpha, \varphi, \gamma)$ let $(\hat{\alpha}, \hat{\varphi}, \hat{\gamma}; i)$ be a minimal Markov-W*-dilation.

If $s\text{-}\lim_{t\to\infty} \gamma_t(x) = \varphi(x)\mathbb{1}$ for all $x \in \alpha$

then $(\hat{\alpha}, \hat{\varphi}, \hat{\gamma}; \hat{\alpha}_0)$ is a generalized K-system.

(cf. [Kü1],3.3.5 Corollary)

Furthermore, if α is finite-dimensional

then the following properties are equivalent:

(a) For all $t>0$, γ_t has only trivial fixed points.

(b) The operator γ_t for $t=1$ has one-dimensional fixed space and 1 is the only eigenvalue of modulus 1 .

(c) $\lim_{t\to\infty} \varphi(x\gamma_t(y)) = \varphi(x)\cdot\varphi(y)$ for all $x,y \in \alpha$.

(d) The expanding W*-system $(\hat{\alpha}, \hat{\varphi}, \hat{\gamma}; \hat{\alpha}_0)$ satisfies the 0-1-property.

(cf. [KÜ1], 3.3.6 Corollary)

Remark

We had results of this type in mind (cf. also the first Corollary in section 3.3.6) when we chose to define a W*-K-system by means of an expanding instead of a contracting W*-subalgebra (cf. 2.2 Remark (1)). In fact, a certain behaviour of the dissipative W*-system $(\mathcal{A}, \varphi, \gamma)$ for $t \to \infty$ is reflected in $(\hat{\mathcal{A}}, \hat{\varphi}, \hat{\gamma})$ by the expanding W*-subalgebra $\hat{\mathcal{A}}_{\mathfrak{F}}$ and not by $\hat{\mathcal{A}}_{\mathfrak{C}}$ which is contracting under $\hat{\gamma}$ in the sense that $\hat{\mathcal{A}}_{\mathfrak{C}} \supseteq \hat{\gamma}_t \, \hat{\mathcal{A}}_{\mathfrak{C}}$ for $t \geq 0$. Rather $\hat{\mathcal{A}}_{\mathfrak{C}}$ is related to the corresponding asymptotic properties of the so called φ-adjoint γ^+ of γ (cf. [KÜ1], in particular 2.2.6 Proposition), i.e. γ^+_t for $t=1$ is the uniquely determined completely positive operator on $\mathcal{A}$ which satisfies

$$\varphi (x \cdot \gamma_t(y)) = \varphi (\gamma^+_t(x) y) \quad \text{for } t=1, x, y \in \mathcal{A} .$$

For illustrating this difference between the expanding subalgebra $\hat{\mathcal{A}}_{\mathfrak{F}}$ and the contracting subalgebra $\hat{\mathcal{A}}_{\mathfrak{C}}$ let us consider the following example:

Example

Let $(\mathcal{M}, \psi)$ be a non-trivial W*-algebra with faithful normal state and let $(\mathcal{A}, \varphi)$ be the infinite tensor product of $\left((\mathcal{M}_n, \psi_n) \right)_{n \in \mathbb{Z}^+}$ with $(\mathcal{M}_n, \psi_n) := (\mathcal{M}, \psi)$ for all $n \in \mathbb{Z}^+$.

Let γ be the representation of $\mathbb{Z}^+$ given by

$\gamma_t(x_1 \otimes \dots \otimes x_n \otimes \mathbb{1} \otimes \dots)$
$= \psi(x_1)\, y_1 \otimes \dots \otimes y_n \otimes \mathbb{1} \otimes \dots$
for $t=1, x_\nu, y_\nu \in \mathcal{M}_\nu$, $y_\nu := x_{\nu+1}$.
Clearly we have $s\text{-}\lim_{t\to\infty} \gamma_t(x) = \varphi(x)\mathbb{1}$ for all $x \in \mathcal{A}$.

On the other hand, the φ-adjoint γ^+ of γ is the right shift on $\mathcal{A}$, i.e.
$\gamma^+_t(x_1 \otimes \dots \otimes x_n \otimes \mathbb{1} \otimes \dots)$
$= \mathbb{1} \otimes y_2 \otimes \dots \otimes y_n \otimes y_{n+1} \otimes \mathbb{1} \otimes \dots$
for $t=1, x_\nu, y_\nu \in \mathcal{M}_\nu, y_{\nu+1} := x_\nu$.
It is not difficult to see that each γ^+_t, $t \in \mathbb{Z}^+$, is an isometry with respect to the norm p_φ on $\mathcal{A}$ (cf. Preliminaries); in particular, it cannot be true that $s\text{-}\lim_{t\to\infty} \gamma^+_t(x) = \varphi(x)\mathbb{1}$ for all $x \in \mathcal{A}$.

Denote by $(\hat{\mathcal{A}}, \hat{\varphi})$ the infinite tensor product of $\big((\mathcal{M}_k, \Psi_k)\big)_{k\in\mathbb{Z}}$ with $(\mathcal{M}_k, \Psi_k) := (\mathcal{M}, \Psi)$ for all $k \in \mathbb{Z}$ and let $\hat{\gamma}$ be the representation of $\mathbb{Z}$ as left shift on $\hat{\mathcal{A}}$. If $i: \mathcal{A} \longrightarrow \hat{\mathcal{A}}$ is the extension of the canonical injection of $\otimes\{\mathcal{M}_n : n \in \mathbb{Z}^+\}$ into $\otimes\{\mathcal{M}_k : k \in \mathbb{Z}\}$ then $(\hat{\mathcal{A}}, \hat{\varphi}, \hat{\gamma}; i)$ forms a minimal Markov-W^*-dilation for $(\mathcal{A}, \varphi, \gamma)$.
Now one verifies that $\hat{\mathcal{A}}_{\mathbb{Q}} = i(\mathcal{A})$ and
$\cap\{\hat{\gamma}_t\, \hat{\mathcal{A}}_{\mathbb{Q}} : t \le 0\} = \mathbb{C}\cdot\mathbb{1}$

On the other hand $\hat{\alpha}_{t_0} = \hat{\alpha}$.

Physically relevant applications of Kümmerer's dilation technique have been worked out in [Kü3], [Kü4]. There we have explicitly constructed various Markov-W^*-dilations for the particular discrete dissipative W^*-system which describes the relaxation of a spin-$\frac{1}{2}$ according to the Bloch equations. The W^*-systems obtained present generalized K-systems which are considerably more complicated than the examples described in this work.
A detailed interpretation of these dilations in the spirit of quantum stochastic processes can be found in [Kü4].

3.3 Quasifree W^*-K-systems over the CAR

The canonical anti-commutation relations (CAR) lead to a functorial relation between Hilbert spaces and operator algebras and so offer a way for constructing W^*-systems with various additional properties starting from Hilbert space constructions.

3.3.1 Let $\mathcal{H}$ be a Hilbert space and $R \in \mathcal{B}(\mathcal{H})$ satisfying $0 \ll R \ll \mathbb{1}$, i.e. R and $\mathbb{1} - R$ are positive definite.

Definition

If S is a unitary representation of $\mathbb{T}$ on $\mathfrak{K}$ such that $R\,S_t = S_t\,R$ for all $t \in \mathbb{T}$ then $(\mathfrak{K}, R, S)$ is called a (discrete or continuous) Hilbert system.

If S is a representation of $\mathbb{T}^+$ as contractions on $\mathfrak{K}$ such that $R\,S_t = S_t\,R$ for all $t \in \mathbb{T}^+$ then $(\mathfrak{K}, R, S)$ is called a (discrete or continuous) dissipative Hilbert system.

Via a representation of the CAR a (dissipative) Hilbert system $(\mathfrak{K}, R, S)$ gives rise to a (dissipative) W^*-system in the following manner (for the terminology and the details of the construction see Appendix) :

Let $A(\mathfrak{K})$ be the CAR-algebra over $\mathfrak{K}$, ω^R the quasifree state on $A(\mathfrak{K})$ corresponding to R and $A^R(S_t)$ for $t \in \mathbb{T}$ or $t \in \mathbb{T}^+$ the quasifree completely positive operator on $A(\mathfrak{K})$ corresponding to S_t via R. If S_t is unitary then $A^R(S_t)$ is an automorphism of $A(\mathfrak{K})$ independent of R.

Define $\mathcal{A}^R(\mathfrak{K})$ as the W^*-algebra generated by the GNS-representation of $A(\mathfrak{K})$ with respect to ω^R and denote the normal extension of ω^R to $\mathcal{A}^R(\mathfrak{K})$ by the same symbol. Let $\mathcal{A}^R(S_t)$ be the normal

extension of $A^R(S_t)$ to $\alpha^R(\mathfrak{A})$ (cf. Appendix). Then $(\alpha^R(\mathfrak{A}), \omega^R, \alpha^R(S))$ forms a (dissipative) W^*-system.

Definition

$(\alpha^R(\mathfrak{A}), \omega^R, \alpha^R(S))$ will be called the quasifree (dissipative) W^*-system corresponding to $(\mathfrak{A}, R, S)$.

For any subset $\mathfrak{K} \subseteq \mathfrak{H}$ denote by $\alpha^R(\mathfrak{K})$ the W^*-subalgebra of $\alpha^R(\mathfrak{A})$ generated by $\{a(f) : f \in \mathfrak{K}\} \subseteq A(\mathfrak{A})$.

3.3.2 Under what conditions on a Hilbert system will the corresponding quasifree W^*-system form a W^*-K-system ?

It seems natural to study the following property.

Definition

Let $(\mathfrak{H}, R, S)$ be a Hilbert system and $\mathfrak{K}$ a closed subspace of $\mathfrak{H}$ such that

(H1) $\mathfrak{K} \subseteq S_t \mathfrak{K}$ for $t \geq 0$,

(H2) $\bigvee \{S_t \mathfrak{K} : t \geq 0\} = \mathfrak{H}$,

(H3) $\bigcap \{S_t \mathfrak{K} : t \leq 0\} = \{0\}$.

Then $(\mathfrak{H}, R, S; \mathfrak{K})$ will be called a Hilbert shift system.

Remark

This name is motivated by the following well-known

fact which will be used several times in the sequel : Given any Hilbert shift system $(\mathcal{H}, R, S; \mathcal{K})$ there is a (possibly non-separable) Hilbert space $\mathcal{L}$ and a Hilbert space isomorphism W from $\mathcal{H}$ onto the Hilbert space tensor product $L^2(\mathbb{T}) \otimes \mathcal{L}$ such that

$W S_t W^* = T_t \otimes \mathbb{1}_{\mathcal{B}(\mathcal{L})}$ and

$W\mathcal{K} = L^2(\mathbb{T}^-) \otimes \mathcal{L}$.

Recall from Remark (2) of 2.2.1 that $L^2(\mathbb{T})$ denotes the Hilbert space of (equivalence classes of) functions on $\mathbb{T}$ which are square integrable with respect to Haar measure on $\mathbb{T}$ and that $L^2(\mathbb{T}^-)$ means the closed subspace of $L^2(\mathbb{T})$ consisting of functions with essential support in $\mathbb{T}^-$. Finally, T is the representation of $\mathbb{T}$ on $L^2(\mathbb{T})$ with

$(T_t f)(s) := f(s-t)$ for $s,t \in \mathbb{T}$, $f \in L^2(\mathbb{T})$.

While the proof of this result is elementary in the discrete case $\mathbb{T} = \mathbb{Z}$, in the continuous case $\mathbb{T} = \mathbb{R}$ it follows from a theorem of von Neumann (cf. for example [Co] , p.457). Remark that it is not necessary here that $\mathcal{H}$ be separable.

This result shows that loosely speaking there is not much choice for $\mathcal{H}$, S and $\mathcal{K}$ subject to the conditions (H1) - (H3). However, even if some non-

trivial Hilbert space $\mathcal{L}$ is given and

$\mathcal{H} = L^2(\mathbb{T}) \otimes \mathcal{L}$,

$S_t = T_t \otimes \mathbb{1}_{\mathcal{B}(\mathcal{L})}$ and

$\mathcal{K} = L^2(\mathbb{T}^-) \otimes \mathcal{L}$

then we are still free in selecting $R \in \{S\}'$ $= \{T\}' \otimes \mathcal{B}(\mathcal{L})$, $0 \ll R \ll \mathbb{1}$. As we shall see in 3.3.4 this freedom will allow us to find a variety of expanding W^*-systems which satisfy the expectation hypothesis or violate it depending on the subset of $\{T\}' \otimes \mathcal{B}(\mathcal{L})$ to which R belongs.

3.3.3 One particular class of Hilbert shift systems we are interested in arises from minimal dilations of dissipative Hilbert systems.
Given a dissipative Hilbert system $(\mathcal{H}, R, S)$ there is a unique Hilbert system $(\hat{\mathcal{H}}, \hat{R}, \hat{S})$ and an isometric embedding I of $\mathcal{H}$ into $\hat{\mathcal{H}}$ such that

$S_t = I^* \hat{S}_t I$ for all $t \geq 0$,

$\hat{\mathcal{H}} = \bigvee \{ \hat{S}_t I \mathcal{H} : t \in \mathbb{T} \}$ and

$\hat{R} I = I R$ (cf. [Em3] , [Ev]) .

Definition

We say that $(\hat{\mathcal{H}}, \hat{R}, \hat{S}; I)$ is the minimal dilation of $(\mathcal{H}, R, S)$.

If $\operatorname{stop-lim}_{t \to \infty} S_t = 0$ then $(\hat{\mathcal{H}}, \hat{R}, \hat{S}; \hat{\mathcal{H}}_0)$ with

$$\mathfrak{K}_{oj} := \bigvee \{ \hat{S}_t \, I \, \mathfrak{K} : t \leq 0 \}$$

is a Hilbert shift system (cf. [Da] , p. 165).

The minimal dilation of $(\mathfrak{K} , R , S)$ immediately provides a minimal Markov-W*-dilation of the corresponding quasifree dissipative W*-system. In fact, $(\alpha^{\hat{R}}(\hat{\mathfrak{K}}) , \omega^{\hat{R}} , \alpha^{\hat{R}}(\hat{S}) ; j)$ with

$j (a(f)) := a(If)$ for $f \in \mathfrak{K}$

forms a minimal Markov-W*-dilation of

$(\alpha^{R}(\mathfrak{K}) , \omega^{R} , \alpha^{R}(S))$.

Definition

We call $(\alpha^{\hat{R}}(\hat{\mathfrak{K}}) , \omega^{\hat{R}} , \alpha^{\hat{R}}(\hat{S}) ; j)$ the quasifree minimal dilation of $(\alpha^{R}(\mathfrak{K}) , \omega^{R} , \alpha^{R}(S))$.

Example

A physically relevant example of a quasifree dilation occurs in the theory of spin-relaxation (cf. [Em4] , [Va]). The so called maximally anisotropic case of the standard Bloch equations for a spin-$\frac{1}{2}$ leads to the continuous quasifree dissipative W*-system corresponding to $(\mathbb{C} , \eta \mathbb{I} , S)$ where $0 < \eta < 1$ and $S_t z := z \exp (- \lambda + i \omega) t$ for $z \in \mathbb{C}$, $t \in \mathbb{R}^+$ with constants $\lambda > 0$, $\omega \in \mathbb{R}$ and i the imaginary unit. Note that $\operatorname{stop-lim}_{t \to \infty} S_t = 0$.

The minimal dilation of $(\mathbb{C} , \eta \mathbb{I} , S)$ and the

corresponding quasifree minimal dilation can be found explicitly in [Va] .

3.3.4 Fix for this subsection a Hilbert shift system $(\mathfrak{K}, R, S; P\mathfrak{K})$ where P is a projection in $\mathcal{B}(\mathfrak{K})$. Recall that $\alpha^R(P\mathfrak{K})$ denotes the W^*-subalgebra of $\alpha^R(\mathfrak{K})$ generated by $A(P\mathfrak{K}) \subseteq A(\mathfrak{K})$ (cf. Appendix or 3.3.1) .
Then the properties (H1) and (H2) of $(\mathfrak{K}, R, S; P\mathfrak{K})$ immediately imply that $(\alpha^R(\mathfrak{K}), \omega^R, \alpha^R(S); \alpha^R(P\mathfrak{K}))$ is an expanding W^*-system.

Definition

Let us call $(\alpha^R(\mathfrak{K}), \omega^R, \alpha^R(S); \alpha^R(P\mathfrak{K}))$ the quasifree expanding W^*-system corresponding to $(\mathfrak{K}, R, S; P\mathfrak{K})$.

We begin our analysis of quasifree expanding W^*-systems by pointing out that for such systems the expectation hypothesis (cf. 2.2.3) is a rather non-trivial condition.

Proposition

The quasifree expanding W^*-system corresponding to $(\mathfrak{K}, R, S; P\mathfrak{K})$ satisfies the expectation hypothesis if and only if $R \in \{S\}' \cap \{P\}'$.

Proof :

As shown in the Appendix the conditional expectation of $(\alpha^R(\mathfrak{A}) , \omega^R)$ onto $\alpha^R(P\mathfrak{A})$ exists if and only if $R P = P R$. q.e.d.

Remarks

(1) By 3.3.2 we may assume that $\{S\}' = \{T\}' \otimes \mathcal{B}(\mathcal{L})$ where $\mathcal{L}$ is a certain Hilbert space, and furthermore $P = K \otimes \mathbb{1}$ where $K \in \mathcal{B}(L^2(\mathbb{T}))$ is the projection onto $L^2(\mathbb{T}^-)$. Hence

$$\{S\}' \cap \{P\}'$$
$$= \left(\{T\}' \otimes \mathcal{B}(\mathcal{L})\right) \cap \left(\{K\}' \otimes \mathcal{B}(\mathcal{L})\right)$$
$$= \left(\{T\}' \cap \{K\}'\right) \otimes \mathcal{B}(\mathcal{L})$$

as a consequence of the commutation theorem for tensor products of von Neumann algebras.

Since $\mathcal{B}(L^2(\mathbb{T}))$ is generated by $\{T\} \cup \{K\}$ we obtain

$$\{S\}' \cap \{P\}' = \mathbb{1} \otimes \mathcal{B}(\mathcal{L}) .$$

(2) If $(\mathfrak{A} , R , S ; P\mathfrak{A})$ arises from a minimal dilation in the manner described in 3.3.3 then it is not difficult to check that $P R = R P$.

(3) It follows from Remark (1) above that $\{S\}'' \cap \{P\}'' = \mathbb{C}\,\mathbb{1}$ because $\{S\}'' = \{T\}'' \otimes \mathbb{1}$.

Hence for all $R \in \{S\}'' \setminus \mathbb{C}\,\mathbb{1}$ the expectation hypothesis fails.

For the class of Hilbert shift systems $(\mathfrak{H}, R, S; P\mathfrak{H})$ satisfying $R \in \{S\}'' \setminus \mathbb{C}\,\mathbb{1}$ we can demonstrate how radically the expectation hypothesis is violated. We begin with a property of $(\mathfrak{H}, R, S; P\mathfrak{H})$.

Lemma

Let $\mathcal{U}$ be a subgroup of the unitary group of $\{S\}''$ such that $\{\mathcal{U}\}'' \neq \mathbb{C}\,\mathbb{1}$. Then

$$\bigvee \{ U\,P\mathfrak{H} : U \in \mathcal{U} \} = \mathfrak{H} .$$

Proof :

By 3.3.2 it suffices to prove the following :

If $\mathcal{U}$ is a subgroup of the unitary group of $\{T\}''$ and $\{\mathcal{U}\}'' \neq \mathbb{C}\,\mathbb{1}$ then

$$\bigvee \{ U\,L^2(\mathbb{T}^-) : U \in \mathcal{U} \} = L^2(\mathbb{T}) .$$

Like above denote by $K \in \mathcal{B}(L^2(\mathbb{T}))$ the projection onto $L^2(\mathbb{T}^-)$ and put $K^\perp := \mathbb{1} - K$. For a non-trivial projection $Q \in \{\mathcal{U}\}''$ we want to show :

$$(*) \quad Q\,L^2(\mathbb{T}) \subseteq \bigvee \{ U\,L^2(\mathbb{T}^-) : U \in \mathcal{U} \} .$$

Since $\mathcal{U}$ is a group the linear hull of $\mathcal{U}$ is already a *-subalgebra which is dense in $\{\mathcal{U}\}''$ with respect to the strong operator topology. Hence Q can be approximated in the strong operator topology by linear combinations of appropriate $U \in \mathcal{U}$, and so the range $R(QK)$ of QK must be in $\bigvee \{ U\,L^2(\mathbb{T}^-) : U \in \mathcal{U} \}$. Since $R(QK) = Q - Q \wedge K^\perp$, for verifying $(*)$ it

is enough to prove $Q \wedge K^{\perp} = 0$.

First observe that $T_t(Q \wedge K^{\perp}) T_t^* \leq Q \wedge K^{\perp}$ for all $t \geq 0$. Suppose that $T_t(Q \wedge K^{\perp}) T_t^* \neq Q \wedge K^{\perp}$ for some some $t > 0$. Then by Beurling's theorem (cf. [He] , p.8 resp. p.39) there is a unitary $V \in \{T\}''$ such that $Q \wedge K^{\perp} = V K^{\perp} V^*$.

This implies $Q \geq V K^{\perp} V^*$ and $Q \geq V T_s K^{\perp} T_s^* V^*$ for all $s \in \overline{\mathbb{T}}$ because $Q , V \in \{T\}'$. Since

$T_s K^{\perp} T_s^* \longrightarrow \mathbb{1}$ for $s \longrightarrow -\infty$

we obtain $Q \geq \mathbb{1}$, a contradiction. So we have verified $Q \wedge K^{\perp} = T_t(Q \wedge K^{\perp}) T_t^*$ for all $t \geq 0$.

It follows that $Q \wedge K^{\perp} \leq T_t K^{\perp} T_t^*$ for all $t \geq 0$.

Now $T_t K^{\perp} T_t^* \longrightarrow 0$ for $t \longrightarrow +\infty$

implies $Q \wedge K^{\perp} = 0$ and hence ($*$) .

Since ($*$) also holds if we replace Q by $\mathbb{1} - Q$ we finally obtain

$$\vee \{ U L^2(\mathbb{T}^-) : U \in \mathcal{U} \} = L^2(\mathbb{T}) \quad . \quad \text{q.e.d.}$$

Proposition

Let σ^R be the modular group associated with $(\alpha^R(\mathfrak{A}) , \omega^R)$. If $R \in \{S\}'' \setminus \mathbb{C}\mathbb{1}$ then

$$\vee \{ \sigma_t^R \alpha^R(P \mathfrak{A}) : t \in \mathbb{R} \} = \alpha^R(\mathfrak{A}) .$$

Proof:

Recall from the Appendix that $\sigma_t^R(a(f)) = a(U_t f)$ with $U_t := R^{it} (\mathbb{1} - R)^{-it}$ for all $t \in \mathbb{R}$, $f \in \mathfrak{A}$.

Now $R \in \{S\}''$ implies $U_t \in \{S\}''$ for all $t \in \mathbb{R}$.
The preceding lemma tells us $V\{U_t P\mathfrak{A} : t \in \mathbb{R}\} = \mathfrak{A}$.
Hence

$$V\{\sigma_t^R \alpha^R(P\mathfrak{A}) : t \in \mathbb{R}\}$$
$$= V\{\alpha^R(U_t P\mathfrak{A}) : t \in \mathbb{R}\}$$
$$= \alpha^R(V\{U_t P\mathfrak{A} : t \in \mathbb{R}\})$$
$$= \alpha^R(\mathfrak{A}) .$$ q.e.d.

In other words, if $R \in \{S\}'' \setminus \mathbb{C}\,\mathbb{I}$ and $\mathfrak{B}$ is a W^*-subalgebra of $\alpha^R(\mathfrak{A})$ such that $\alpha^R(P\mathfrak{A}) \subseteq \mathfrak{B}$ and the conditional expectation of $(\alpha^R(\mathfrak{A}), \omega^R)$ onto $\mathfrak{B}$ exists then $\mathfrak{B} = \alpha^R(\mathfrak{A})$.

3.3.5 For our further analysis of quasifree expanding W^*-systems it will be useful to introduce a concept which is quite natural when one thinks of stochastic processes.

Definition

Let $(\mathfrak{A}, \varphi, \alpha; \mathfrak{A}_{0]})$ be an expanding W^*-system and assume that there is another W^*-subalgebra $\mathfrak{A}_{[0}$ of $\mathfrak{A}$ such that $(\mathfrak{A}, \varphi, \alpha^{-1}; \mathfrak{A}_{[0})$ with the representation $\alpha^{-1}: \mathbb{T} \longrightarrow \mathrm{Aut}(\mathfrak{A}, \varphi)$, $t \longmapsto \alpha^{-1}_t := \alpha_{-t}$ is an expanding W^*-system, too. Define

$\mathfrak{A}_{t]} := \alpha_t(\mathfrak{A}_{0]})$,
$\mathfrak{A}_{[t} := \alpha_t(\mathfrak{A}_{[0})$ and

$\alpha_{[s,t]} := \alpha_{[s} \cap \alpha_{t]}$ for all $s, t \in \mathbb{T}$, $s \leq t$.

If $\alpha_{0]}$ and $\alpha_{[0}$ are related such that

(T1) $\alpha_{0]} = \bigvee \{ \alpha_{[s,0]} : s \leq 0 \}$ and

(T2) $\alpha_{[0} = \bigvee \{ \alpha_{[0,t]} : t \geq 0 \}$

then $(\alpha, \varphi, \alpha; \alpha_{0]}, \alpha_{[0})$ will be called a two-sided expanding W*-system.

Examples

(1) Let $(\hat{\alpha}, \hat{\varphi}, \hat{\gamma}; i)$ be a minimal W*-dilation of a dissipative W -system $(\alpha, \varphi, \gamma)$.

Obviously, $(\hat{\alpha}, \hat{\varphi}, \hat{\gamma}; \hat{\alpha}_{0]})$ and $(\hat{\alpha}, \hat{\varphi}, \hat{\gamma}^{-1}; \hat{\alpha}_{[0})$ as defined in 3.2 are expanding W*-systems.

For any $t \in \mathbb{T}$ we have

$\hat{\alpha}_{t]} = \bigvee \{ \hat{\gamma}_s \circ i (\alpha) : s \leq t \}$ and

$\hat{\alpha}_{[t} = \bigvee \{ \hat{\gamma}_s \circ i (\alpha) : s \geq t \}$ and so for $s, t \in \mathbb{T}$

$\hat{\alpha}_{[s,t]} \supseteq \bigvee \{ \hat{\gamma}_r \circ i (\alpha) : r \in [s,t] \}$.

This implies

$$\begin{aligned} &\bigvee \{ \hat{\alpha}_{[s,0]} : s \leq 0 \} \\ &\supseteq \bigvee \{ \bigvee \{ \hat{\gamma}_r \circ i (\alpha) : r \in [s,0] \} : s \leq 0 \} \\ &= \bigvee \{ \hat{\gamma}_s \circ i (\alpha) : s \leq 0 \} \\ &= \hat{\alpha}_{0]} . \end{aligned}$$

As the reverse inclusion is trivial we have proved (T1). Likewise one obtains (T2).

Hence $(\hat{\alpha}, \hat{\varphi}, \hat{\gamma}; \hat{\alpha}_{0]}, \hat{\alpha}_{[0})$ is a two-sided expanding W*-system.

(2) Examples of a different kind arise from Hilbert shift systems. Let $(\mathfrak{A}, R, S ; P\mathfrak{A})$ be a Hilbert shift system and $P^{\perp} := \mathbb{1} - P$. The properties (H1) and (H3) of $(\mathfrak{A}, R, S ; P\mathfrak{A})$ (cf. 3.3.2) imply that $(\alpha^R(\mathfrak{A}), \omega^R, \alpha^R(S)^{-1} ; \alpha^R(P^{\perp}\mathfrak{A}))$ is an expanding W^*-system, too. For (T1) observe that $P\mathfrak{A} = \bigvee\{ S_t P^{\perp}\mathfrak{A} \cap P\mathfrak{A} : t \leq 0 \}$.
Hence $A(P\mathfrak{A})$ is the norm closure of $\bigcup\{ A(S_t P^{\perp}\mathfrak{A} \cap P\mathfrak{A}) : t \leq 0 \}$. Since

$$\bigvee\{ \alpha^R(S_t P^{\perp}\mathfrak{A}) \cap \alpha^R(P\mathfrak{A}) : t \leq 0 \}$$
$$\geq \bigvee\{ A(S_t P^{\perp}\mathfrak{A}) \cap A(P\mathfrak{A}) : t \leq 0 \}$$
$$\geq \bigvee\{ A(S_t P^{\perp}\mathfrak{A} \cap P\mathfrak{A}) : t \leq 0 \}$$

we conclude (T1).
Likewise (T2) follows.
Hence $(\alpha^R(\mathfrak{A}), \omega^R, \alpha^R(S) ; \alpha^R(P\mathfrak{A}), \alpha^R(P^{\perp}\mathfrak{A}))$ forms a two-sided expanding W^*-system.

3.3.6 Now we turn to proving that a quasifree expanding W^*-system corresponding to a Hilbert shift system in fact always satisfies the 0-1-property.
Studying a class of dilations constructed in a quasifree manner over the canonical commutation relations (CCR) Emch, Albeverio and Eckmann [Em3] verify the 0-1-property by an argument of the following kind :
Suppose $(\alpha, \varphi, \alpha ; \alpha_{0]}, \alpha_{[0})$ is a two-sided

expanding W^*-system and $\mathcal{A}_{0]}$ is in the commutant of $\mathcal{A}_{[0}$ then it is not difficult to see that $\bigcap\{\mathcal{A}_{t]} : t \leq 0\}$ is contained in the center of $\mathcal{A}$. Hence $(\mathcal{A}, \varphi, \alpha; \mathcal{A}_{0]})$ satisfies the 0-1-property provided $\mathcal{A}$ is a factor.

On p.83 of [Em3] the authors remark that Fermi systems can be treated in an analogous way - "with the proviso that one works with the observable-algebra rather than the field-algebra" (for the terminology see the Appendix) . This idea is carried out in example 5 of [Em1] .

For dealing with expanding W^*-systems $(\mathcal{A}^R(\mathfrak{X}), \omega^R, \alpha^R(S); \alpha^R(P\mathfrak{X}))$ where $\mathcal{A}^R(\mathfrak{X})$ is of course the full field-algebra we employ a method which is based on adapting ideas from the theory of quasilocal algebras (cf. [Br1], in particular p.123) .

Theorem

Let $(\mathcal{A}, \varphi, \alpha; \mathcal{A}_{0]}, \mathcal{A}_{[0})$ be a two-sided expanding W^*-system, E a conditional expectation of $(\mathcal{A}, \varphi)$ commuting with α, and $F := Id - E$. Suppose that for all $x \in \mathcal{A}_{0]}$, $y \in \mathcal{A}_{[0}$ we have

(A1) $E(x)\,y = y\,E(x)$ and

(A2) $F(x)\,F(y) = -\,F(y)\,F(x)$.

Then $\bigcap\{\mathcal{A}_{t]} : t \leq 0\}$ is contained in the center of $\mathcal{A}$.

Proof :

Given a self-adjoint $x \in \bigcap \{ \mathcal{A}_{t]} : t \leq 0 \}$, $\|x\| = 1$, we want to conclude $F(x) = x - E(x) = 0$: By (T1) (cf. 3.3.5) for each $t \in \mathbb{T}^-$ the *-algebra $\bigcup \{ \mathcal{A}_{[s,t]} : s < t \}$ is $s(\mathcal{A}, \mathcal{A}_*)$-dense in $\mathcal{A}_{t]}$. Therefore by Kaplansky's density theorem we can choose for each $n \in \mathbb{N}$ an element $t_n \in \mathbb{T}^-$, $t_n < -n$, a self-adjoint $x_n \in \mathcal{A}^1_{[t_n,-n]}$ such that $s\text{-}\lim_{n\to\infty} x_n = x$. Then the s-continuity of F yields

$$F(x) = s\text{-}\lim_{n\to\infty} F(x_n) .$$

Since multiplication is jointly s-continuous on the unit ball of $\mathcal{A}$ we conclude

$$F(x)^2 = s\text{-}\lim_{n\to\infty} s\text{-}\lim_{m\to\infty} F(x_n) F(x_m) .$$

Now for $n \in \mathbb{N}$ fixed we have $F(x_n) \in F(\mathcal{A}_{[t_n,-n]})$, and so for any $m \in \mathbb{N}$, $-m < t_n$, we obtain

$$F(x_n) F(x_m) = - F(x_m) F(x_n)$$

by property (A2) . Therefore

$$\begin{aligned} F(x)^2 &= s\text{-}\lim_{n\to\infty} \Big(- s\text{-}\lim_{m\to\infty} F(x_m) F(x_n) \Big) \\ &= - s\text{-}\lim_{n\to\infty} F(x) F(x_n) \\ &= - F(x)^2 \end{aligned}$$

which implies $F(x) = 0$.

This means $\bigcap \{ \mathcal{A}_{t]} : t \leq 0 \} = \bigcap \{ E(\mathcal{A}_{t]}) : t \leq 0 \}$. Next property (A1) entails that $E(\mathcal{A}_{t]})$ is in

the relative commutant $(\alpha_{[t})^{\mathcal{C}}$ of $\alpha_{[t}$ for all $t \in \mathbb{T}^-$. Since $\bigcap \{ (\alpha_{[t})^{\mathcal{C}} : t \neq 0 \}$ is the center of α we have proved the assertion. q.e.d.

Corollary

Let $(\mathfrak{X}, R, S ; P\mathfrak{X})$ be a Hilbert shift system. Then the corresponding quasifree expanding W^*-system satisfies the 0-1-property.

Proof :

As we have seen in 3.3.5 $(\alpha^R(\mathfrak{X}), \omega^R, \alpha^R(S) ; \alpha^R(P\mathfrak{X}), \alpha^R(P^{\perp}\mathfrak{X}))$ with $P^{\perp} := \mathbb{1} - P$ forms a two-sided expanding W^*-system. On $\alpha^R(\mathfrak{X})$ define an automorphism ε by $\varepsilon(a(f)) := - a(f)$ for $f \in \mathfrak{X}$ and put $E := \frac{1}{2}(Id + \varepsilon)$.

Then the CAR entail that the conditions (A1), (A2) of the preceding Theorem are satisfied. So the oberservation that $\alpha^R(\mathfrak{X})$ is a factor [Po] finishes the proof. q.e.d.

Definition

Let us call $(\alpha^R(\mathfrak{X}), \omega^R, \alpha^R(S) ; \alpha^R(P\mathfrak{X}))$ the quasifree W^*-K-system corresponding to $(\mathfrak{X}, R, S ; P\mathfrak{X})$.

Combined with 3.3.4 the Corollary above provides a

wealth of W^*-K-systems violating the expectation hypothesis. Of particular interest is the following situation.

Example

Let $(\mathfrak{A}, R, S ; P\mathfrak{A})$ be a continuous Hilbert shift system with $R = (\mathbb{1} + e^{-H})^{-1}$ where H denotes the self-adjoint infinitesimal generator of S. Then $S_t = e^{itH} = R^{it}(\mathbb{1} - R)^{-it}$ for all $t \in \mathbb{R}$ so that $\alpha^R(S)$ equals the modular automorphism group σ^R associated with $(\alpha^R(\mathfrak{A}), \omega^R)$ (cf. Appendix).

Thus we have here the first examples where a modular automorphism group forms a W^*-K-system.

As a further consequence we obtain a result about quasifree minimal dilations corresponding to what Emch, Albeverio and Eckmann [Em3] have shown for the CCR case.

Corollary

Let $(\alpha^R(\mathfrak{A}), \omega^R, \alpha^R(S))$ be a quasifree dissipative W^*-system and $(\alpha^{\hat{R}}(\hat{\mathfrak{A}}), \omega^{\hat{R}}, \alpha^{\hat{R}}(\hat{S}) ; i)$ its quasifree minimal dilation (cf. 3.3.3).

If $\text{stop-lim}_{t \to \infty} S_t = 0$ then $(\alpha^{\hat{R}}(\hat{\mathfrak{A}}), \omega^{\hat{R}}, \alpha^{\hat{R}}(\hat{S}) ; \alpha^{\hat{R}}(\hat{\mathfrak{A}})_0)$ forms a generalized K-system.

Recall that $\alpha^{\hat{R}}(\hat{\mathfrak{A}})_{0]} = \bigvee\{\alpha^{\hat{R}}(\hat{S}_t)\circ i(\alpha^{\hat{R}}(\mathfrak{A})) : t \leq 0\}$ as defined in 3.2 .

Proof :

If $(\hat{\mathfrak{A}}, \hat{R}, \hat{S}; I)$ is the minimal dilation of $(\mathfrak{A}, R, S)$ then stop-lim $S_t = 0$ implies that $(\hat{\mathfrak{A}}, \hat{R}, \hat{S}; \hat{\mathfrak{A}}_{0]})$ with $\hat{\mathfrak{A}}_{0]} = \bigvee\{\hat{S}_t \circ I\mathfrak{A} : t \leq 0\}$ is a Hilbert shift system. So the assertion is a consequence of the preceding Corollary when we take into account that

$$\alpha^{\hat{R}}(\hat{\mathfrak{A}})_{0]} = \bigvee\{\alpha^{\hat{R}}(\hat{S}_t \circ I\mathfrak{A}) : t \leq 0\}$$
$$= \alpha^{\hat{R}}(\bigvee\{\hat{S}_t \circ I\mathfrak{A} : t \leq 0\}) = \alpha^{\hat{R}}(\hat{\mathfrak{A}}_{0]})$$

and that the projection from $\hat{\mathfrak{A}}$ onto $\hat{\mathfrak{A}}_{0]}$ commutes with $\hat{R}$. q.e.d.

Example

The quasifree minimal dilation of the maximally anisotropic case of the standard Bloch equation for a spin-$\frac{1}{2}$ (cf. 3.3.3) forms a generalized K-system (cf. [Kü 3]) .

3.3.7 In a certain respect the quasifree W^*-K-systems differ from commutative K-systems more than all the generalized K-systems constructed by Emch et al. [Em1] , [Em3] .

Every example of a generalized K-system described in

[Em1], [Em3] and of course every W^*-K-system $(\mathcal{A}, \varphi, \alpha; \mathcal{A}_0)$ with commutative $\mathcal{A}$ satisfies the following property :

Definition (cf. [Em1])

Let $(\mathcal{A}, \varphi, \alpha; \mathcal{A}_0)$ be an expanding W^*-system and denote by $\mathcal{A}_0^c$ the relative commutant of $\mathcal{A}_0$ in $\mathcal{A}$. If $(\mathcal{A}, \varphi, \alpha^{-1}; \mathcal{A}_0^c)$ is an expanding W^*-system, too, then $(\mathcal{A}, \varphi, \alpha; \mathcal{A}_0)$ is called weakly reversible.

This property implies a rather strong form of asymptotic abelianness and by that simplifies the structure theory of $\mathcal{A}$ considerably (cf. [Em1], 2.Theorem and subsequent Remarks). Using Emch's result G.A.Raggio [Ra] proved that a weakly reversible W^*-K-system is s-asymptotically abelian.

Definition

A W^*-system $(\mathcal{A}, \varphi, \alpha)$ is called s-asymptotically abelian if $\operatorname{s-lim}_{|t|\to\infty} [\alpha_t(x), y] = 0$ for all $x, y \in \mathcal{A}$

Here we use the notation $[x_1, x_2] := x_1x_2 - x_2x_1$ for $x_1, x_2 \in \mathcal{A}$.

Remark

The following two observations may indicate the strength of this property :

(1) If a W^*-system $(\mathcal{A}, \varphi, \alpha)$ is asymptotically abelian in the norm topology then $\mathcal{A}$ is commutative [G1] .

(2) On the other hand, $(\mathcal{A}, \varphi, \alpha)$ is asymptotically abelian with respect to $\sigma(\mathcal{A}, \mathcal{A}_*)$ if $(\mathcal{A}, \varphi, \alpha)$ is strongly mixing, i.e. if for all $x, y \in \mathcal{A}$

$$\lim_{|t|\to\infty} \varphi(\alpha_t(x) y) = \varphi(x) \varphi(y) .$$

Now let us see that the quasifree W^*-system corresponding to a Hilbert system $(\mathfrak{H}, R, S)$ cannot be s-asymptotically abelian :

For $f \in \mathfrak{H}$, $f \neq 0$, and $x_t := [a(S_t f), a(f)]$ we have $\omega^R(x_t^* x_t) = 4 \langle Rf | f \rangle^2 - 4 | \langle Rf | S_t f \rangle |^2$.

Since the spectrum of S is absolutely continuous (cf. [Da] , p.163) the Riemann-Lebesgue Lemma implies

$$\lim_{|t|\to\infty} \omega^R(x_t^* x_t)^{1/2} = 2 \langle Rf | f \rangle \neq 0 .$$

Hence $(\mathcal{A}^R(\mathfrak{H}), \omega^R, \alpha^R(S))$ is not s-asymptotically abelian.

In particular, the quasifree W^*-K-systems provide the first examples of W^*-K-systems which are not weakly reversible.

4 Elementary operations

4.1 Direct product

Let $(\mathcal{A}, \varphi, \alpha)$ and $(\mathcal{B}, \psi, \beta)$ be two W^*-systems, both discrete or both continuous.

On the W^*-tensor product $\mathcal{A} \otimes \mathcal{B}$ define a representation $\alpha \otimes \beta$ of $\mathbb{T}$ in $\mathrm{Aut}(\mathcal{A} \otimes \mathcal{B}, \varphi \otimes \psi)$ by

$(\alpha \otimes \beta)_t := \alpha_t \otimes \beta_t$.

Note that the normal state $\varphi \otimes \psi$ on $\mathcal{A} \otimes \mathcal{B}$ is faithful (cf. [Ta2], p.227).

Definition

By analogy to the classical theory (cf. [Co], p.227) the W^*-system $(\mathcal{A} \otimes \mathcal{B}, \varphi \otimes \psi, \alpha \otimes \beta)$ is called the direct product of $(\mathcal{A}, \varphi, \alpha)$ and $(\mathcal{B}, \psi, \beta)$.

Proposition

Let $(\mathcal{A}, \varphi, \alpha; \mathcal{A}_\mathfrak{J})$ and $(\mathcal{B}, \psi, \beta; \mathcal{B}_\mathfrak{J})$ be two expanding W^*-systems, both discrete or both continuous.

Then $(\mathcal{A} \otimes \mathcal{B}, \varphi \otimes \psi, \alpha \otimes \beta; \mathcal{A}_\mathfrak{J} \otimes \mathcal{B}_\mathfrak{J})$ is an expanding W^*-system, too.

It is a W^*-K-system if and only if both components are.

Furthermore, it satisfies the expectation property if and only if both components do so.

Proof:

A corollary, [Ta2] , p.227, of the commutation theorem for tensor products of von Neumann algebras tells us

$$\vee\{\alpha_{t]} \otimes \mathcal{B}_{t]} : t \geq 0\}$$
$$= \vee\{\alpha_{t]} : t \geq 0\} \otimes \vee\{\mathcal{B}_{t]} : t \geq 0\}$$
$$= \alpha \otimes \mathcal{B}$$

and

$$\cap\{\alpha_{t]} \otimes \mathcal{B}_{t]} : t \leq 0\}$$
$$= \cap\{\alpha_{t]} : t \leq 0\} \otimes \cap\{\mathcal{B}_{t]} : t \leq 0\}$$

Hence the direct product is an expanding W^*-system and furthermore satisfies the 0-1-property if and only if both components do so.

Suppose the conditional expectation E resp. F of (α, φ) onto $\alpha_{0]}$ resp. of $(\mathcal{B}, \psi)$ onto $\mathcal{B}_{0]}$ exists. Then $E \otimes F$ is the conditional expectation of $(\alpha \otimes \mathcal{B}, \varphi \otimes \psi)$ onto $\alpha_{0]} \otimes \mathcal{B}_{0]}$ (cf. [Ta2] , p.228).

Assume to the contrary that the conditional expectation P of $(\alpha \otimes \mathcal{B}, \varphi \otimes \psi)$ onto $\alpha_{0]} \otimes \mathcal{B}_{0]}$ exists. Let Q be the conditional expectation of $(\alpha_{0]} \otimes \mathcal{B}_{0]}, \varphi \otimes \psi)$ onto $\alpha_{0]} \otimes \mathbb{1}$. Then the restriction of $Q \circ P$ to $\alpha \otimes \mathbb{1}$ can be identified with the conditional expectation of (α, φ) onto $\alpha_{0]}$. q.e.d.

Remark

By this result we can amplify our collection of W^*-K-systems violating the expectation hypothesis.

4.2 Restriction to an invariant subalgebra

Proposition

Let $(\mathcal{A}, \varphi, \alpha; \mathcal{A}_{0]})$ be a W^*-K-system and $\mathcal{B}$ a W^*-subalgebra of $\mathcal{A}$ globally invariant under α. Suppose the conditional expectation E of $(\mathcal{A}, \varphi)$ onto $\mathcal{B}$ exists and satisfies $E(\mathcal{A}_{0]}) \subseteq \mathcal{A}_{0]}$. Then $(\mathcal{B}, \varphi\restriction\mathcal{B}, \alpha\restriction\mathcal{B}; \mathcal{A}_{0]} \cap \mathcal{B})$ is again a W^*-K-system.

It satisfies the expectation hypothesis if $(\mathcal{A}, \varphi, \alpha; \mathcal{A}_{0]})$ does.

Proof :

Put $\mathcal{B}_{t]} := \alpha_t(\mathcal{A}_{0]} \cap \mathcal{B})$ for $t \in \mathbb{T}$. Since $\mathcal{B}_{t]} = \mathcal{A}_{t]} \cap \mathcal{B}$ the properties (K1) and (K3) are easy to verify. For (K2) observe that $\alpha_t \circ E \circ \alpha_{-t} = E$ for all $t \in \mathbb{T}$ because $\mathcal{B}$ is globally invariant under α and the conditional expectation of $(\mathcal{A}, \varphi)$ onto $\mathcal{B}$ is unique. This implies $E(\mathcal{A}_{t]}) \subseteq \mathcal{B}_{t]}$ for all $t \in \mathbb{T}$ and furthermore $E(\bigcup\{\mathcal{A}_{t]} : t \geq 0\}) \cup \{\mathcal{B}_{t]} : t \geq 0\}$.

Finally the weak* continuity of E entails

$$\mathcal{B} = E(\vee\{\alpha_{t]} : t \geq 0\}) \subseteq \vee\{\mathcal{B}_{t]} : t \geq 0\} \subseteq \mathcal{B}$$

which proves (K2) .

If F is the conditional expectation of $(\mathcal{A}, \varphi)$ onto $\mathcal{A}_{0]}$ then $E \cdot F \restriction \mathcal{B}$ is the conditional expectation of $(\mathcal{B}, \varphi \restriction \mathcal{B})$ onto $\mathcal{B}_{0]}$. q.e.d.

Corollary

Let $(\mathcal{A}, \varphi, \alpha ; \mathcal{A}_{0]})$ be a W*-K-system and $\mathcal{G}$ a semigroup of completely positive contractions on $\mathcal{A}$ satisfying the following properties :

(1) $\varphi \circ \gamma = \varphi$,

(2) $\gamma \cdot \alpha_t = \alpha_t \circ \gamma$ and

(3) $\gamma(\mathcal{A}_{0]}) \subseteq \mathcal{A}_{0]}$ for all $t \in \mathbb{T}$, $\gamma \in \mathcal{G}$.

Put $\mathcal{F} := \{x \in \mathcal{A} : \gamma(x) = x \text{ for all } \gamma \in \mathcal{G}\}$ and $\mathcal{F}_{0]} := \mathcal{A}_{0]} \cap \mathcal{F}$.

Then $(\mathcal{F}, \varphi \restriction \mathcal{F}, \alpha \restriction \mathcal{F} ; \mathcal{F}_{0]})$ is a W*-K-system.

Proof :

Assumption (2) implies that $\mathcal{F}$ is globally invariant under α . According to [KÜ2] the weak* mean ergodic projection P of $\mathcal{G}$ is the conditional expectation of $(\mathcal{A}, \varphi)$ onto $\mathcal{F}$.

Finally, from assumption (3) we infer $P(\mathcal{A}_{0]}) \subseteq \mathcal{A}_{0]}$. q.e.d.

Remark

The preceding Proposition presents a W^*-algebraic version of the classical operation of considering a factor of a measure theoretical dynamical system (cf. [Co] , p.230, [Wa] , p.61).
The Proposition and its Corollary extend ideas of G.G.Emch, cf. 8.Theorem and the subsequent Remarks in [Em1] .

Example

Applying the Corollary above to quasifree W^*-K-systems we can easily obtain W^*-K-systems which are built over the observable-algebra in a CAR-algebra (cf. Appendix). Generalized K-systems of such a kind are alluded to in the discussion of [Em3] , p.83 (cf. also 3.3.6).
Given a quasifree W^*-K-system $(\alpha^R(\mathfrak{K}) , \omega^R , \alpha^R(S) ; \alpha^R(P\mathfrak{K}))$ let $\sigma^R(\mathfrak{K})$ resp. $\sigma^R(P\mathfrak{K})$ be the W^*-subalgebra of $\alpha^R(\mathfrak{K})$ generated by the observable-algebra in $A(\mathfrak{K})$ resp. in $A(P\mathfrak{K})$ (cf. Appendix).
If we define

$\gamma_z(a(f)) := a(zf)$ for all $z \in \mathbb{T}$, $|z| = 1$, $f \in \mathfrak{K}$,

then $\{ \gamma_z : z \in \mathbb{T} , |z| = 1 \} \subseteq \mathrm{Aut}(\alpha^R(\mathfrak{K}))$

meets all the requirements of the Corollary and its fix algebra is $\sigma^R(\mathfrak{K})$.

Hence $(\mathfrak{G}^R(\mathfrak{A}), \omega^R \restriction \mathfrak{G}^R(\mathfrak{A}), \alpha^R(s) \restriction \mathfrak{G}^R(\mathfrak{A}); \mathfrak{G}^R(P\mathfrak{A}))$ is a W^*-K-system.

A generalized K-system of a similar kind can be found in example 5 of [Em1] .

5 Homogeneous Lebesgue spectrum

As suggested by the work of Sinai [Si2] one may approach the analysis of a W^*-K-system via the concept of homogeneous Lebesgue spectrum (cf. 2.2.3) .

Definition (cf. [Si2])

A unitary representation V of $\mathbb{T}$ on a Hilbert space $\mathcal{K}$ is said to have homogeneous Lebesgue spectrum if there is another Hilbert space $\mathcal{L}$ and a Hilbert space isomorphism W from $\mathcal{K}$ onto the Hilbert space tensor product $L^2(\mathbb{T}) \otimes \mathcal{L}$ such that

$W V_t W^* = T_t \otimes 1\!\!1_{\mathcal{B}(\mathcal{L})}$ for all $t \in \mathbb{T}$

with $(T_t f)(s) := f(s-t)$ for $f \in L^2(\mathbb{T})$, $s \in \mathbb{T}$.

In other words, V is conjugate to a shift on $L^2(\mathbb{T}, \mathcal{L})$.

For a motivation of Sinai's terminology we refer to [Co] , p.456 where for the discrete case $\mathbb{T} = \mathbb{Z}$ this property is formulated in terms of spectral measures being equivalent to Lebesgue measure.

For the whole of this chapter let $(\mathfrak{A}, \varphi, \alpha; \mathfrak{A}_0)$ be an expanding W^*-system and $(\mathfrak{H}, \xi, U)$ the GNS-triple associated with $(\mathfrak{A}, \varphi, \alpha)$ (cf. Preliminaries). The following notation will be used frequently : If $\mathcal{M}$ is a subset of $\mathfrak{H}$ we denote by $\overline{\mathcal{M}}$ the closure of $\mathcal{M}$ in $\mathfrak{H}$.

5.1 Homogeneous Lebesgue spectrum on the GNS-Hilbert space

Motivated by the classical theory (cf. 2.2.3) and by the result that any Hilbert shift system has homogeneous Lebesgue spectrum (cf. 3.3.2) we first consider the following question :

Suppose $(\alpha , \varphi , \alpha ; \alpha_{0]})$ satisfies the 0-1-property.

If we put $\mathcal{H}_{0]} := \overline{\alpha_{0]} \xi}$ does it follow that

$\mathcal{H}_{0]} \subseteq U_t \mathcal{H}_{0]}$ for all $t \geq 0$,

$\vee \{ U_t \mathcal{H}_{0]} : t \geq 0 \} = \mathcal{H}$, and

$\cap \{ U_t \mathcal{H}_{0]} : t \leq 0 \} = \mathbb{C} \xi$?

Of course, only the verification of the property

(P) $\cap \{ U_t \mathcal{H}_{0]} : t \leq 0 \} = \mathbb{C} \xi$

is critical.

5.1.1 Actually, it turns out that property (P) does not hold for all W*-K-systems. This phenomenon comes out most drastically in the case when α equals the modular automorphism group associated with (α , φ) . Recall that this is for example the case for certain quasifree W*-K-systems (cf. 3.3.6) .

Proposition

Let $(\alpha , \varphi , \alpha ; \alpha_{0]})$ be an expanding W*-system such that α is the modular automorphism group

associated with $(\mathcal{A}, \varphi)$.

Then $\mathcal{H}_{[0]} = \overline{\mathcal{A}_{[0]}\xi} = \mathcal{H}$.

Proof :

Recall that $\alpha_t(x)\xi = \Delta^{it} x\xi$ for all $x \in \mathcal{A}$, $t \in \mathbb{T}$ where Δ denotes the modular operator associated with the pair $(\mathcal{A}, \xi)$ (cf. [Br1] , p.89).

We show that the orthogonal complement $(\mathcal{A}_{[0]}\xi)^{\perp}$ of $\mathcal{A}_{[0]}\xi$ in $\mathcal{H}$ is trivial.

Because of (K2) it suffices to verify that

$(\mathcal{A}_{[0]}\xi)^{\perp} \subseteq (\cup\{ \Delta^{it} \mathcal{A}_0 \xi : t \geq 0 \})^{\perp}$.

Now let $x \in \mathcal{A}_{[0]}$, $s > 0$ and $\eta \in (\mathcal{A}_{[0]}\xi)^{\perp}$ be given.

Since $x\xi$ is in the domain of $\Delta^{1/2}$ the definition

$F(z) := \langle \Delta^{z} x\xi | \eta \rangle$

gives a continuous bounded function F on $\mathcal{S} := \{ z \in \mathbb{C} : 0 \leq \mathrm{Re}\ z \leq \frac{1}{2} \}$ which is holomorphic on the interior of $\mathcal{S}$ (cf. [St] , p.214) .

For $t \leq 0$ we know

$F(it) = \langle \Delta^{it} x\xi | \eta \rangle = 0$

because $\alpha_t(x) \in \mathcal{A}_{[0]}$ according to (K1) .

An application of the Schwarz reflection principle shows $F(z) = 0$ for all $z \in \mathcal{S}$, $\mathrm{Im}\ z \leq 0$, hence F vanishes identically.

In particular, $F(is) = \langle \Delta^{is} x\xi | \eta \rangle = 0$. q.e.d.

Remark

If $(\mathcal{A}, \varphi, \alpha; \mathcal{A}_{0]})$ satisfies the 0-1-property and α equals the modular automorphism group associated with $(\mathcal{A}, \varphi)$ then we have just seen that $\overline{\mathcal{A}_{t]}\xi} = \overline{\mathcal{A}\xi} = \mathcal{H}$ for all $t \in \mathbb{T}$ though $\bigcap \{\mathcal{A}_{t]} : t \neq 0\} = \mathbb{C}\mathbb{1}$.

So we have here examples of a situation that has elicited considerable interest (cf. [Di] and references therein) : While the W^*-subalgebra $\mathcal{A}_{t]}$ of $\mathcal{A}$ is arbitrarily small with regard to set inclusion the vector $\xi \in \mathcal{H}$ is still cyclic for $\mathcal{A}_{t]}$.

5.1.2 In some sense contrary to the situation of the foregoing Proposition is the case when the modular automorphism group associated with $(\mathcal{A}, \varphi)$ leaves $\mathcal{A}_{0]}$ globally invariant or, equivalently, when $(\mathcal{A}, \varphi, \alpha; \mathcal{A}_{0]})$ satisfies the expectation hypothesis.

Actually, for generalized K-systems G.G.Emch [Em1] has shown that property (P) does hold. It is not difficult to see that his proof applies to any W^*-K-system $(\mathcal{A}, \varphi, \alpha; \mathcal{A}_{0]})$ provided that for each $t \in \mathbb{T}^-$ the conditional expectation E_t of $(\mathcal{A}_{0]}, \varphi)$ onto $\mathcal{A}_{t]}$ exists. Then property (P) follows by a "decreasing martingale lemma" [Em1] which

states that for all $x \in \alpha_{0]}$ the net $(E_t(x))_{t \in \mathbb{T}^-}$ converges to $\varphi(x)\mathbb{1}$ in $s(\alpha , \alpha_*)$.

Definition

Let $(\mathcal{M} , \psi)$ be a W^*-algebra with faithful normal sate and $(\mathcal{M}_\nu)_{\nu \in I}$ a decreasing net of W^*-subalgebras of $\mathcal{M}$ such that for each $\nu \in I$ the conditional expectation E_ν of $(\mathcal{M} , \psi)$ onto $\mathcal{M}_\nu$ exists.
Then $(E_\nu)_{\nu \in I}$ is called the (decreasing) martingale of $(\mathcal{M} , \psi)$ adapted to $(\mathcal{M}_\nu)_{\nu \in I}$.

With this terminology the reasoning outlined above can be summarized as follows : The W^*-K-system $(\alpha , \varphi , \alpha ; \alpha_{0]})$ satisfies property (P) if the martingale of $(\alpha_{0]} , \varphi \restriction \alpha_{0]})$ adapted to $(\alpha_{t]})_{t \in \mathbb{T}^-}$ exists.
Thus it seems interesting to look for criteria for the existence of such a martingale.

Proposition

The following are equivalent :

(a) The martingale of $(\alpha_{0]} , \varphi \restriction \alpha_{0]})$ adapted to $(\alpha_{t]})_{t \in \mathbb{T}^-}$ exists.

(b) For some $s , t \in \mathbb{T}$, $s < t$, the conditional expectation of $(\alpha_{t]} , \varphi \restriction \alpha_{t]})$ onto $\alpha_{s]}$ exists.

(c) $(\alpha , \varphi , \alpha ; \alpha_{0]})$ satisfies the expectation hypothesis.

(d) The martingale of (α , φ) adapted to $(\alpha_{t]})_{t \in \mathbb{T}}$ exists.

Proof :

(c) implies that the modular automorphism group σ associated with (α , φ) leaves $\alpha_{0]}$, hence any $\alpha_{t]}$ globally invariant because σ commutes with α . So we conclude (d) .

From this (a) follows by restriction to $\alpha_{0]}$.

Since (a) implies (b) trivially we are finished when we have shown (b) $\Rightarrow$ (c) :

Given $s , t \in \mathbb{T}$, $s < t$, and the conditional expectation E of $(\alpha_{t]}, \varphi \restriction \alpha_{t]})$ onto $\alpha_{s]}$ then $E_1 := \alpha_{-s} \circ E \circ \alpha_s$ is the conditional expectation of $(\alpha_{r]}, \varphi \restriction \alpha_{r]})$ onto $\alpha_{0]}$ for $r := t-s > 0$.

Define recursively $E_{n+1} := E_n \circ (\alpha_{nr} \circ E_1 \circ \alpha_{-nr})$ for for all $n \in \mathbb{N}$.

Then E_n is the conditional expectation of $(\alpha_{nr]}, \varphi \restriction \alpha_{nr]})$ onto $\alpha_{0]}$ and we have $E_n \restriction \alpha_{mr]} = E_m$ for $m \leq n$

by the uniqueness of the conditional expectation

Hence we can consistently define a map P from $\alpha_\infty := \bigcup \{ \alpha_{t]} : t \in \mathbb{T} \} = \bigcup \{ \alpha_{nr]} : n \in \mathbb{N} \}$ onto $\alpha_{0]}$ by putting

$P(x) := E_n (x)$ if $x \in \alpha_{nr]}$.

P is a projection of norm 1 such that $\varphi \circ P = \varphi$.

Furthermore, P is a contraction with respect to the norm p_φ (cf. Preliminaries). Hence P has a unique p_φ-continuous extension $\bar{P}$ to the closure $\mathcal{A}$ of $\mathcal{A}_\infty$ with respect to p_φ. It satisfies $\bar{P}^2 = \bar{P}$ and $\varphi \circ \bar{P} = \varphi$. Finally, $\bar{P}$ has norm 1 (with respect to the norm on $\mathcal{A}$) since the closure of $\mathcal{A}_\infty \cap \mathcal{A}^1$ with respect to p_φ is $\mathcal{A}^1$.
Hence $\bar{P}$ is the conditional expectation of $(\mathcal{A}, \varphi)$ onto $\mathcal{A}_{0]}$. q.e.d.

5.2 Homogeneous Lebesgue spectrum on the domain of $\Delta^{1/2}$

While we have seen in 5.1.2 that property (P) holds true if the modular automorphism group σ associated with $(\mathcal{A}, \varphi)$ leaves $\mathcal{A}_{0]}$ globally invariant, we recall from 5.1.1 that (P) definitely fails in the case $\alpha = \sigma$. This indicates that we should modify the approach of 5.1 and take into account the modular operator Δ associated with the pair $(\mathcal{A}, \xi)$.
Instead of the Hilbert space $(\mathcal{H}, \langle . \mid . \rangle)$ let us consider the domain $\mathcal{D}$ of $\Delta^{1/2}$ equipped with the inner product $\langle . \mid . \rangle_\#$ defined by

$$\langle x\xi \mid y\xi \rangle_\# := \varphi(y^*x) + \varphi(xy^*)$$
$$= \langle x\xi \mid y\xi \rangle + \langle \Delta^{1/2} x\xi \mid \Delta^{1/2} y\xi \rangle$$

for all $x, y \in \mathcal{A}$.

The corresponding norm

$$\begin{aligned}\|x\xi\|_{\#} &= (\varphi(x^*x) + \varphi(xx^*))^{1/2} \\ &= (\|x\xi\|^2 + \|x^*\xi\|^2)^{1/2} \\ &= (p_\varphi(x)^2 + p_\varphi(x^*)^2)^{1/2} \quad , \ x \in \mathfrak{A} \ ,\end{aligned}$$

is the graph norm of $\Delta^{1/2}$, and $\mathfrak{D}$ is the completion of $\mathfrak{A}\xi$ with respect to $\|\cdot\|_{\#}$.

One can show that the norms $x\xi \longmapsto \|x\xi\|$ and $x\xi \longmapsto \|x\xi\|_{\#}$ on $\mathfrak{A}\xi$ are equivalent if and only if there is a constant $c \geq 1$ and a trace τ such that $c^{-1}\tau \leq \varphi \leq c\,\tau$.

The following observations will be useful in the sequel.

Remarks

(1) $\varphi(x) = \frac{1}{2}\langle x\xi \mid \xi\rangle_{\#}$ for all $x \in \mathfrak{A}$.

(2) The orthogonal projection P_ξ from $\mathfrak{D}$ onto $\mathbb{C}\xi$ is given by $P_\xi\eta = \frac{1}{2}\langle \eta \mid \xi\rangle_{\#}\,\xi$

Note that $\|\xi\|_{\#} = 2$.

(3) Given $x \in \mathfrak{A}$, we have a densely defined operator

$$\mathfrak{A}\xi \longrightarrow \mathfrak{D} \ , \ z\xi \longmapsto xz\xi$$

which is in general not bounded with respect to $\|\cdot\|_{\#}$.

However, the linear functional

$$\mathfrak{A}\xi \longrightarrow \mathbb{C} \ , \ z\xi \longmapsto \langle xz\xi \mid \xi\rangle_{\#}$$

is bounded with respect to $\|\cdot\|_{\#}$:

$$\begin{aligned}|\langle xz\xi \mid \xi\rangle_{\#}| &\leq 2\,|\langle xz\xi \mid \xi\rangle| \\ &\leq 2\,\|z\xi\|\cdot\|x^*\xi\| \\ &\leq 2\,\|x^*\|\,\|z\xi\|_{\#} \ .\end{aligned}$$

(4) The norm $\mathcal{A} \longrightarrow \mathbb{R}$, $x \longmapsto \|x\xi\|_{\#}$ generates a topology on $\mathcal{A}$ which is weaker than $s^*(\mathcal{A}, \mathcal{A}_*)$, but on $\mathcal{A}^1$ coincides with $s^*(\mathcal{A}, \mathcal{A}_*)$ (cf. [Ta2], p.148).

(5) We have $\mathfrak{D} = \overline{\mathcal{A}^{sa}\xi} + i\,\overline{\mathcal{A}^{sa}\xi}$ and $\overline{\mathcal{A}^{sa}\xi} \cap i\,\overline{\mathcal{A}^{sa}\xi} = \{0\}$ where i denotes as usual the imaginary unit, $i^2 = 1$ [Ri].

(6) The unitary representation U of $\mathbb{T}$ which implements α in the GNS-representation (cf. 5.1) leaves $\mathfrak{D}$ globally invariant, and its restriction to $(\mathfrak{D}, \langle . \,|\, . \rangle_{\#})$ is again a unitary representation because each U_t , $t \in \mathbb{T}$, commutes with $\Delta^{1/2}$. We denote this restriction again by U.

For $\mathcal{M} \subseteq \mathfrak{D}$ we shall denote by $\overline{\mathcal{M}}^{\#}$ the closure of $\mathcal{M}$ with respect to $\|.\|_{\#}$.

Now we are ready to carry out the program outlined in the introduction of 5.1 - with the modification that $\mathfrak{A}$ is replaced by $\mathfrak{D}$.

Theorem

If $(\mathcal{A}, \varphi, \alpha; \mathcal{A}_0)$ is a W*-K-system then for $\mathfrak{D}_0 := \overline{\mathcal{A}_0\xi}^{\#}$ we have

(D1) $\mathfrak{D}_0 \subseteq U_t\,\mathfrak{D}_0$ for all $t \geq 0$,

(D2) $\vee\{U_t\,\mathfrak{D}_0 : t \geq 0\} = \mathfrak{D}$,

(D3) $\cap\{U_t\,\mathfrak{D}_0 : t \leq 0\} = \mathbb{C}\xi$.

In particular, the unitary representation U on $\mathfrak{D}$ has homogeneous Lebesgue spectrum on the orthogonal complement of $\mathbb{C}\xi$ in $(\mathfrak{D}, \langle . \mid . \rangle_{\#})$.

Proof :

By Remark (4) above the space $\bigcup \{ \alpha_{t]} \xi : t \geq 0 \}$ is dense in $\alpha\xi$ with respect to $\| . \|_{\#}$ because $\bigcup \{ \alpha_{t]} : t \geq 0 \}$ is dense in α with respect to $s^*(\alpha , \alpha_*)$. This implies (D2) .

If we observe that $U_t \mathfrak{D}_{0]} = \overline{\alpha_{t]}^{sa} \xi} + i \overline{\alpha_{t]}^{sa} \xi}$ with $\overline{\alpha_{t]}^{sa} \xi} \cap i \overline{\alpha_{t]}^{sa} \xi} = \{ 0 \}$ the property (D3) is an immediate consequence of the following Lemma.

q.e.d.

Lemma

Let $(\mathfrak{M}_\nu)_{\nu \in I}$ be a decreasing net of von Neumann algebras on a Hilbert space $\mathfrak{K}$ with common separating vector $x \in \mathfrak{K}$. Then

$$\bigcap \{ \overline{\mathfrak{M}_\nu^{sa} x} : \nu \in I \} = \overline{\bigcap \{ \mathfrak{M}_\nu : \nu \in I \}^{sa} x} .$$

Proof :

In order to apply a result of Rieffel and van Daele [Ri] (cf.also [Ta2] , p.224) we consider $\mathfrak{K}$ as a real Hilbert space by restricting scalar multiplication to $\mathbb{R}$ and by taking the real part of the scalar product $\langle . \mid . \rangle$.

So if $\mathfrak{k}$ is a real subspace of $\mathfrak{K}$, define

$$\mathcal{L}^{\perp} := \{\eta \in \mathcal{K} : \operatorname{Re}\langle \eta \mid \lambda \rangle = 0 \text{ for all } \lambda \in \mathcal{L}\}.$$

Since $\chi \in \mathcal{K}$ is cyclic for each commutant $\mathcal{M}_\nu'$ we have by [Ri] :

$(\mathcal{M}_\nu'^{sa}\chi)^{\perp} = i\,\overline{\mathcal{M}_\nu^{sa}\chi}$ for each $\nu \in I$.

In the following we use a simplified notation and for example write $\bigcap_\nu \mathcal{M}_\nu$ instead of $\bigcap\{\mathcal{M}_\nu : \nu \in I\}$. Due to the fact that $\bigcup_\nu \mathcal{M}_\nu'$ is a strongly dense $*$-subalgebra of $\bigvee_\nu \mathcal{M}_\nu'$ and consequently $(\bigcup_\nu \mathcal{M}_\nu')^{sa}$ is strongly dense in $(\bigvee_\nu \mathcal{M}_\nu')^{sa}$ we have

$$\overline{\bigcup_\nu \mathcal{M}_\nu'^{sa}\chi} = \overline{(\bigcup_\nu \mathcal{M}_\nu')^{sa}\chi} = \overline{(\bigvee_\nu \mathcal{M}_\nu')^{sa}\chi}.$$

Combining the two observations we obtain

$$\begin{aligned}\bigcap_\nu \overline{\mathcal{M}_\nu^{sa}\chi} &= i\bigcap_\nu((\mathcal{M}_\nu'^{sa}\chi)^{\perp}) \\ &= i\,(\bigcup_\nu \mathcal{M}_\nu'^{sa}\chi)^{\perp} \\ &= i\,((\bigcap_\nu \mathcal{M}_\nu)'^{sa}\chi)^{\perp} \\ &= \overline{(\bigcap_\nu \mathcal{M}_\nu)^{sa}\chi}\end{aligned}$$

having applied [Ri] another time. q.e.d.

6 Mixing properties

In this last chapter we turn to analysing the asymptotic behaviour of a W^*-K-system.

6.1 The 0-1-property as a mixing condition

First the results of section 5.2 will enable us to give a straightforward proof for known equivalences between the 0-1-property and certain rather strong mixing conditions. In a somewhat different form these mixing conditions play an important role in the theory of quasilocal algebras (cf. [Br1] , p. 118) . We give here a new proof which efficiently uses the Hilbert space $(\mathfrak{D} , \langle . \mid . \rangle_*)$ introduced above.

Theorem (cf. [Br1] , p. 118)

Let $(\mathcal{A} , \varphi)$ be a W^*-algebra with faithful normal state and $(\mathcal{A}_\nu)_{\nu \in I}$ a decreasing net of W^*-subalgebras of $\mathcal{A}$.

Then the following are equivalent:

(a) $\bigcap \{ \mathcal{A}_\nu : \nu \in I \} = \mathbb{C}\, 1\!\!1$

(b) $\lim\limits_{\nu \in I} \sup \{ \mid \varphi(xy) - \varphi(x)\varphi(y) \mid : y \in \mathcal{A}_\nu^1 \} = 0$ for all $x \in \mathcal{A}$.

(c) $\lim\limits_{\nu \in I} \sup \{ \mid \varphi(xz) - \varphi(x)\varphi(z) \mid : z \in \mathcal{A}_\nu^{sa}, \varphi(z^2) \leq 1 \} = 0$ for all $x \in \mathcal{A}$.

Proof:

For showing (a) $\Rightarrow$ (c) let ($\mathfrak{H}, \xi$) be the GNS-pair associated with ($\mathfrak{A}$, φ) (cf. Preliminaries) and let $\mathfrak{D}$ be the completion of $\mathfrak{A}\xi \subseteq \mathfrak{H}$ with respect to the scalar product

$$\langle x\xi \mid y\xi \rangle_{\#} := \langle x\xi \mid y\xi \rangle + \langle y^*\xi \mid x^*\xi \rangle$$

for $x , y \in \mathfrak{A}$.

Denote by Q_ν , $\nu \in I$, the orthogonal projection of $(\mathfrak{D}, \langle . \mid . \rangle_{\#})$ onto $\overline{\mathfrak{A}_\nu \xi}^{\#}$. Then the Lemma of 5.2 implies that the net $(Q_\nu)_{\nu \in I}$ converges strongly in $\mathfrak{B}(\mathfrak{D})$ to the orthogonal projection P_ξ onto $\mathbb{C}\xi$.

Now let $x \in \mathfrak{A}$ and $\varepsilon > 0$ be given. According to Remark (3) of 5.2 there is $\eta \in \mathfrak{D}$ such that

$$\langle xy\xi \mid \xi \rangle_{\#} = \langle y\xi \mid \eta \rangle_{\#} \quad \text{for all } y \in \mathfrak{A} .$$

Next choose $\mu \in I$ such that $\| (Q_\mu - P_\xi)\eta \| \leq \varepsilon\sqrt{2}$.

Since $Q_\mu y\xi = y\xi$ for all $y \in \mathfrak{A}_\mu$ we have

$$\begin{aligned}
&\langle xy\xi \mid \xi \rangle_{\#} - \tfrac{1}{2} \langle x\xi \mid \xi \rangle_{\#} \cdot \langle y\xi \mid \xi \rangle_{\#} \\
&= \langle y\xi \mid \eta \rangle_{\#} - \tfrac{1}{2} \langle y\xi \mid \xi \rangle_{\#} \cdot \langle \xi \mid \eta \rangle_{\#} \\
&= \langle y\xi \mid Q_\mu \eta - \tfrac{1}{2} \langle \eta \mid \xi \rangle_{\#} \xi \rangle_{\#} \\
&= \langle y\xi \mid (Q_\mu - P_\xi)\eta \rangle_{\#} .
\end{aligned}$$

Using the equations

$\varphi(ab) = \frac{1}{2} \langle ab\xi \mid \xi \rangle_{\#}$ for all $a , b \in \mathfrak{A}$ and

$\varphi(a^2) = \frac{1}{2} \| a\xi \|_{\#}^2$ for all $a \in \mathfrak{A}^{sa}$

we obtain

$$\begin{aligned} & | \varphi(xy) - \varphi(x) \varphi(y) | \\ & \leq \frac{1}{2} \| y\xi \|_{\#} \cdot \| (Q_{\mu} - P_{\xi}) \eta \|_{\#} \\ & \leq \frac{1}{\sqrt{2}} \varphi(y^2)^{1/2} \cdot \sqrt{2}\, \varepsilon \\ & = \varepsilon \varphi(y^2)^{1/2} . \end{aligned}$$

Hence

$\sup \{ | \varphi(xy) - \varphi(x)\varphi(y) | : y \in \mathfrak{A}_{\mu}^{sa}, \varphi(y^2) \leq 1 \} \leq \varepsilon$

which proves (c) .

For verifying (c) $\Rightarrow$ (b) fix $x \in \mathfrak{A}$. If $y \in \mathfrak{A}$ is given and $y = y_1 + iy_2$ is its decomposition into real and imaginary part then $\max(\| y_1 \| , \| y_2 \|) \leq \| y \|$ implies

$$\begin{aligned} & \sup \{ | \varphi(xy) - \varphi(x)\varphi(y) | : y \in \mathfrak{A}_{[1]}^{1} \} \\ & \leq 2 \sup \{ | \varphi(xz) - \varphi(x)\varphi(z) | : z \in \mathfrak{A}_{[1]}^{1} \cap \mathfrak{A}^{sa} \} \\ & \leq 2 \sup \{ | \varphi(xz) - \varphi(x)\varphi(z) | : z \in \mathfrak{A}_{[1]}^{sa} , \varphi(z^2) \leq 1 \} \end{aligned}$$

where the last estimate follows from the fact that

$$\varphi(z^2) \leq \| z \|^2 \qquad \text{for all } z \in \mathcal{A}^{sa}.$$

Finally we prove the implication (b) $\Rightarrow$ (a) : If (a) is false we can choose $y \in \bigcap \{ \mathcal{A}_\nu : \nu \in I \}$ satisfying $\| y \| \leq 1$ and $y \neq \varphi(y)\, \mathbb{1}$. Then there is $x \in \mathcal{A}$ such that $\varphi(xy) \neq \varphi(x)\varphi(y)$, hence

$$\lim_{\nu \in I} \sup \{ | \varphi(xz) - \varphi(x)\varphi(z) | : z \in \mathcal{A}_\nu^1 \}$$

$$\geq | \varphi(xy) - \varphi(x)\varphi(y) | \neq 0$$

contradicting (b). q.e.d.

Property (b) of the foregoing Theorem can be reformulated in various ways. For simplifying some of the following statements we shall denote by $\mathcal{A}^P$ the set of all projections of a given W^*-algebra $\mathcal{A}$.

Proposition

Let $(\mathcal{A}, \varphi)$ be a W^*-algebra with faithful normal state and $(\mathcal{A}_\nu)_{\nu \in I}$ a net of W^*-subalgebras of $\mathcal{A}$. Then the following are equivalent:

(a) $\lim_{\nu \in I} \| (\psi - \varphi) \restriction \mathcal{A}_\nu \| = 0$ for all states $\psi \in \mathcal{A}_*$.

(b) $\lim_{\nu \in I} \sup \{ | \varphi(xy) - \varphi(x)\varphi(y) | : y \in \mathcal{A}_\nu^1 \} = 0$ for all $x \in \mathcal{A}$.

(c) $\lim\limits_{\nu\in I} \sup \{ |\varphi(pq) - \varphi(p)\varphi(q)| : q \in \mathfrak{A}_\nu^P \} = 0$
for all $p \in \mathfrak{A}^P$.

(d) $\lim\limits_{\nu\in I} \sup \{ |\varphi(x^*yx) - \varphi(x^*x)\varphi(y)| : y \in \mathfrak{A}_\nu^1 \} = 0$
for all $x \in \mathfrak{A}$.

(e) $\lim\limits_{\nu\in I} \sup \{ |\varphi(pqp) - \varphi(p)\varphi(q)| : q \in \mathfrak{A}_\nu^P \} = 0$
for all $p \in \mathfrak{A}^P$.

Proof :

Clearly, (á) is equivalent to

(a') $\lim\limits_{\nu\in I} \| (\mu - \mu(\Pi)\varphi) \restriction \mathfrak{A}_\nu \| = 0$ for all $\mu \in \mathfrak{A}_*$.

Given $x \in \mathfrak{A}$ consider (a') for $L_x\varphi$ with
$(L_x\varphi)(y) := \varphi(xy)$ for all $y \in \mathfrak{A}$.
This leads to (b) which implies (c) trivially.
For the implication (c) $\Rightarrow$ (a') note that for
any $\mu \in \mathfrak{A}_*$ we have

$$\| \mu \restriction \mathfrak{A}_\nu \| \leq 4 \sup \{ |\mu(y)| : y \in \mathfrak{A}_\nu^1 \cap \mathfrak{A}_\nu^+ \}$$
$$= 4 \sup \{ |\mu(q)| : q \in \mathfrak{A}_\nu^P \}$$

because $\mathfrak{A}_\nu^1 \cap \mathfrak{A}_\nu^+$ is the weak*-closed convex hull
of $\mathfrak{A}^P$.
Hence (c) implies
$\lim\limits_{\nu\in I} \| (L_p\varphi - \varphi(p)\varphi) \restriction \mathfrak{A}_\nu \| = 0$ for any $p \in \mathfrak{A}^P$.
Since $\{ L_x\varphi : x \in \mathfrak{A} \}$ is norm-dense in $\mathfrak{A}_*$,
$\{ L_p\varphi : p \in \mathfrak{A}^P \}$ is norm-total in $\mathfrak{A}_*$. So (a')
follows.

While (a) $\Rightarrow$ (d) $\Rightarrow$ (e) is trivial the implication (e) $\Rightarrow$ (a) is a consequence of the following Lemma. q.e.d.

Lemma

Let $(\mathfrak{A}, \varphi)$ be a W^*-algebra with faithful normal state and denote by $\mathfrak{A}_*^+$ the positive part of $\mathfrak{A}_*$. Define a map $M : \mathfrak{A}^+ \longrightarrow \mathfrak{A}_*^+$ by

$(Mx)(y) := \varphi(xyx)$ for $x \in \mathfrak{A}^+$, $y \in \mathfrak{A}$.

Then $M\mathfrak{A}^P$ is norm-total in $\mathfrak{A}_*$.

Proof :

By a Radon-Nikodym theorem (cf. [Sa] , p.76) we have

$M\mathfrak{A}^+ \supseteq \mathbb{R}^+ \{ \psi \in \mathfrak{A}_*^+ : \psi \leq \varphi \}$.

As φ is faithful, $\mathbb{R}^+ \{ \psi \in \mathfrak{A}_*^+ : \psi \leq \varphi \}$ and hence $M\mathfrak{A}^+$ is norm-dense in $\mathfrak{A}_*^+$.

Furthermore, M is norm-continuous :

$$\| Mx - My \| \leq (\|x\| + \|y\|) \| x - y \|$$

for all $x , y \in \mathfrak{A}^+$.

Thus for proving the Lemma it suffices to find a norm-dense subset of $\mathfrak{A}^+$ which is mapped by M into the linear hull of $M\mathfrak{A}^P$.

Let $\mathfrak{I} \subseteq \mathfrak{A}^+$ be the set of all finite linear combinations $x = \sum_n \lambda_n p_n$ of pairwise orthogonal $p_n \in \mathfrak{A}^P$ with coefficients $\lambda_n \in \mathbb{R}^+$.

Given such an element $x \in \mathfrak{d}$, for all $y \in \mathfrak{A}$ we have

$$xyx = \sum_n \lambda_n^2 \, p_n y p_n + \sum_{m<n} \lambda_m \lambda_n \left[p_m y p_n + p_n y p_m \right] .$$

Since

$$p_m y p_n + p_n y p_m$$

$$= (p_m + p_n) y (p_m + p_n) - p_m y p_m - p_n y p_n$$

we obtain

$$Mx = \sum_n \lambda_n^2 \, M p_n + \sum_{m<n} \lambda_m \lambda_n \left[M(p_m + p_n) - M p_m - M p_n \right] .$$

As $p_m + p_n \in \mathfrak{A}^P$ for $m \neq n$ we see that Mx is in the linear hull of $M\mathfrak{A}^P$.

Finally, by the spectral theorem it is clear that $\mathfrak{d}$ is norm-dense in $\mathfrak{A}^+$. q.e.d.

Corollary

An expanding W*-system $(\mathfrak{A}, \varphi, \alpha ; \mathfrak{A}_0)$ satisfies the 0-1-property if and only if any of the following conditions holds :

(a) For every $x \in \mathfrak{A}$ we have

$$\sup \left\{ \left| \varphi(\alpha_t(x) z) - \varphi(x) \varphi(z) \right| : z \in \mathfrak{A}_0^{sa}, \ \varphi(z^2) \leq 1 \right\} \longrightarrow 0$$

as $t \longrightarrow +\infty$.

(b) For every state $\psi \in \mathfrak{A}_*$ we have

$$\left\| (\alpha_{t*}(\psi) - \varphi) \restriction \mathfrak{A}_0 \right\| \longrightarrow 0 \quad \text{as} \quad t \longrightarrow +\infty .$$

Here α_{t*}, $t \in \mathbb{T}$, denotes the pre-adjoint of α_t.

Proof :

The equivalence of the 0-1-property with (a) is an immediate consequence of the foregoing Theorem since

$$\sup \{ |\varphi(\alpha_t(x)z) - \varphi(x)\varphi(z)| : z \in \mathcal{A}_{0]}^{sa}, \varphi(z^2) \leq 1 \}$$

$$= \sup \{ |\varphi(xz) - \varphi(x)\varphi(z)| : z \in \mathcal{A}_{-t]}^{sa}, \varphi(z^2) \leq 1 \}$$

due to $\varphi(\alpha_t(x)z) = \varphi(x\,\alpha_{-t}(z))$.

For the equivalence with (b) we simply note that

$$\| (\alpha_{t*}(\psi) - \varphi) \restriction \mathcal{A}_{0]} \| = \| (\psi - \varphi) \restriction \mathcal{A}_{-t]} \|$$

for all $t \in \mathbb{T}^+$. q.e.d.

6.2 K-mixing

The conditions (b) - (e) of the Proposition in 6.1 resemble more or less a property called "K-mixing" which is fruitfully employed in classical ergodic theory (cf. [Co] , in particular p.283).

6.2.1 Let (Ω, Σ, μ, T) be a measure theoretical dynamical system which for simplicity we assume to be discrete. Given a σ-subalgebra Ξ of Σ define

$$\Xi_{k]} := \bigvee \{ T_n \Xi : n \leq k \} .$$

Definition (cf. [Co] , p.26)

(Ω, Σ, μ, T) is called K-mixing if for any set $A \in \Sigma$ and any finite subalgebra Ξ of Σ we have

$$\lim_{k \to -\infty} \sup \{ | \mu(A \cap B) - \mu(A)\mu(B) | : B \in \Xi_{k]} \} = 0.$$

K-mixing is of importance for the theory of K-systems because of the following result (cf. [Co] , p.283) : Provided (Ω, Σ, μ) is a Lebesgue space, (Ω, Σ, μ, T) can be equipped with the structure of a K-system if and only if it is K-mixing.

6.2.2 The possibilities of a non-commutative extension of K-mixing have apparently not been studied yet. The following generalization might seem reasonable :

Definition

Let $(\mathfrak{M}, \psi, \tau)$ be a discrete W^*-system.

If $\mathfrak{N}$ is a W^*-subalgebra of $\mathfrak{M}$ put

$$\mathfrak{N}_{k]} := \bigvee \{ \tau_n(\mathfrak{N}) : n \leq k \} .$$

Then $(\mathfrak{M}, \psi, \tau)$ is called W^*-K-mixing if for any $x \in \mathfrak{M}$ and any finite-dimensional subalgebra $\mathfrak{N}$ of $\mathfrak{M}$ we have

$$\lim_{k \to -\infty} \sup \{ | \varphi(xy) - \varphi(x)\varphi(y) | : y \in \mathfrak{N}^1_{k]} \} = 0 .$$

Remarks

(1) Applying the results of 6.1 one sees that there are various reformulations of W^*-K-mixing. Those expressed in terms of projections are immediate analogues of the classical K-mixing.

(2) A disputable feature of our definition may be the use of $V\{\tau_n(\mathfrak{N}) : n \leq k\}$ which can hardly be controlled when $\mathfrak{M}$ is non-commutative. For example, the hyperfinite factor of type II_1 is generated by two commutative finite-dimensional subalgebras [Be].

Of course, our interest in the concept of W^*-K-mixing concerns the question whether under suitable conditions there are connections between W^*-K-mixing and W^*-K-systems analogous to the classical theory. One such condition replacing the assumption that (Ω, Σ, μ) is a Lebesgue space might be the requirement that the predual of the W^*-algebra under consideration is separable.

Easily one proves the following :

Proposition

Let $(\alpha, \varphi, \varkappa; \alpha_0]$ be a discrete W^*-K-system and $\mathfrak{B}$ a finite-dimensional subalgebra contained in $\alpha_\infty := \bigcup\{\alpha_n] : n \geq 0\}$. Then for all $x \in \alpha$:

$$\lim_{k \to -\infty} \sup\{|\varphi(xy) - \varphi(x)\varphi(y)| : y \in \mathfrak{B}_{1_k]}\} = 0 \quad .$$

Proof :

There is $m \in \mathbb{N}$ such that $\mathfrak{b} \subseteq \mathfrak{A}_{m]}$. Hence $\mathfrak{b}_{k]} \subseteq \mathfrak{A}_{m+k]}$ for all $k \in \mathbb{Z}$ and so $\bigcap \{ \mathfrak{b}_{k]} : k \leq 0 \} \subseteq \{ \mathfrak{A}_{m+k]} : k \leq 0 \} = \mathbb{C} \mathbb{1}$.
By the Theorem in 6.1 the assertion follows.

q.e.d.

However, the assertion of the Proposition is no longer true if we drop the restriction that $\mathfrak{b}$ is contained in $\mathfrak{A}_\infty$! In fact, there are discrete generalized K-systems $(\mathfrak{A} , \varphi , \alpha ; \mathfrak{A}_{0]})$ with separable $\mathfrak{A}_*$ which are not W^*-K-mixing.

Example

Let T be the shift representation of $\mathbb{Z}$ on $\mathfrak{H} := L^2(\mathbb{T})$ (cf. 3.3.2) , let $E \in \{ T \}'$ be a non-trivial projection, $f \in E\mathfrak{H}$ a unit vector and $\mathcal{F} := \bigvee \{ T_k f : k \leq 0 \}$.
Clearly, $T_1 \mathcal{F} \supseteq \mathcal{F}$.
If $T_1 \mathcal{F} \neq \mathcal{F}$ Beurling's Theorem [He] provides a unitary $W \in \{ T \}''$ such that $\mathcal{F} = W \mathfrak{H}_{0]}$ with $\mathfrak{H}_{0]} := L^2(\mathbb{T}^-)$.
This implies $\bigvee \{ T_n \mathcal{F} : n \geq 0 \} = \mathfrak{H}$ in contradiction to $\bigvee \{ T_n \mathcal{F} : n \geq 0 \} \subseteq E\mathfrak{H} \neq \mathfrak{H}$.
Hence $T_1 \mathcal{F} = \mathcal{F}$.

Note that $\mathcal{F} \cap \bigcup \{ T_n \mathfrak{H}_{0]} : n \geq 0 \} = \{ 0 \}$.

This is due to $\mathcal{F} \cap \mathcal{H}_{\Omega} = \{0\}$ which one verifies by the same arguments as in the proof of the Lemma in 3.3.4 .

Now for an arbitrary $R \in \{T\}'$, $0 \ll R \ll \mathbb{1}$, let $(\alpha^R(\mathcal{A}) , \omega^R , \alpha^R(T) ; \alpha^R(\mathcal{A}_{\Omega}))$ be the quasifree W^*-K-system corresponding to $(\mathcal{H}, R , T ; \mathcal{H}_{\Omega})$ (cf. 3.3.6) . Observe that $\alpha^R(\mathcal{A})_*$ is separable (cf. Appendix).

Let $\mathcal{B}$ be the four-dimensional subalgebra of $\alpha^R(\mathcal{A})$ generated by $a(f)$. Then $\mathcal{B}_{\Omega}$ equals the W^*-subalgebra $\alpha^R(\mathcal{F})$ of $\alpha^R(\mathcal{A})$ generated by $\{ a(h) : h \in \mathcal{F} \} \subseteq A(\mathcal{A})$ .Indeed, since $\{ T_k f : k \leq 0 \}$ is total in $\mathcal{F}$, the W^*-subalgebra $\alpha^R(\mathcal{F})$ is also generated by $\{ a(T_k f) : k \leq 0 \} \subseteq \mathcal{B}_{\Omega}$.

By $\mathcal{B}_{\Omega} = \alpha^R(\mathcal{F})$ we see that $\mathcal{B}_{\Omega}$ is globally invariant under $\alpha^R(T)$ because $\mathcal{F}$ is globally invariant under T .

In particular, $(\alpha^R(\mathcal{A}) , \omega^R, \alpha^R(T) ; \alpha^R(\mathcal{A}_{\Omega}))$ cannot be W^*-K-mixing.

6.3 Mixing of arbitrary degree

In the asymptotic analysis of a classical K-system one considers a property called mixing of arbitrary degree

(cf. [Co] , p.25 ; [Si1] , [Si2]) . It is an immediate consequence of K-mixing.

6.3.1 As a non-commutative generalization of that mixing property we consider the following concept:

Definition

A W^*-system $(\mathfrak{A}, \varphi, \alpha)$ is called W^*-mixing of arbitrary degree if for every $n \in \mathbb{N}$ the following holds:

Given $x_0, \ldots, x_n, y_0, \ldots, y_n \in \mathfrak{A}$ and $\varepsilon > 0$ there is $m \in \mathbb{N}$ such that

$$\left| \varphi(x_0 \alpha_{t_1}(x_1) \ldots \alpha_{t_1+\ldots+t_n}(x_n y_n) \ldots \alpha_{t_1}(y_1) y_0) - \prod_{\nu=0}^{n} \varphi(x_\nu y_\nu) \right| < \varepsilon$$

provided that $t_\nu \leq -m$ for all $\nu \in \{1, \ldots, n\}$.

For short we write

$$\lim_{t_1, \ldots, t_n \to -\infty} \varphi(x_0 \alpha_{t_1}(x_1) \ldots \alpha_{t_1+\ldots+t_n}(x_n y_n) \ldots \alpha_{t_1}(y_1) y_0) = \prod_{\nu=0}^{n} \varphi(x_\nu y_\nu) .$$

Remarks

(1) If $(\mathfrak{A}, \varphi, \alpha)$ arises from a measure theoretical dynamical system as described in 2.2.1 then W^*-mixing of arbitrary degree is obviously equivalent to the usual mixing of arbitrary degree (cf. [Co] , p.25) .

(2) In the theory of quasilocal algebras one can find

a different non-commutative version of mixing of arbitrary degree (cf. [La] , 2.4 Corollary , p.198) . The concept of Lanford and Ruelle is formulated in a C*-algebraic setting and uses the richer structure of a quasilocal algebra. A W*-algebraic extension of the classical result that every K-system is mixing of arbitrary degree seems not to be available in the literature.

(3) Clearly, W*-mixing of arbitrary degree is equivalent to the following:

For all $n \in \mathbb{N}$ and $x_1, \ldots, x_n, y_1, \ldots, y_n \in \mathfrak{A}$ we have in the $\sigma(\mathfrak{A}, \mathfrak{A}_*)$-topology:

$$\lim_{t_1,\ldots,t_n \to -\infty} \alpha_{t_1}(x_1) \ldots \alpha_{t_1+\ldots+t_n}(x_n y_n) \ldots \alpha_{t_1}(y_1)$$

$$= \prod_{\nu=1}^{n} \varphi(x_\nu y_\nu)\, \mathbb{1} \ .$$

(4) Expressions of the form

$$\varphi(x_0 \alpha_{t_1}(x_1) \ldots \alpha_{t_1+\ldots+t_n}(x_n y_n) \ldots \alpha_{t_1}(y_1) y_0)$$

are studied in the theory of non-commutative stochastic processes under the name of "correlation kernels" (cf. [Ac2]) .

W*-mixing of arbitrary degree implies another mixing property describing asymptotic behaviour for $(t_1, \ldots, t_n) \longrightarrow +\infty$.

Proposition

If $(\mathcal{A}, \varphi, \alpha)$ is W^*-mixing of arbitrary degree then for every $n \in \mathbb{N}$ the following holds :

Given $z_0, \ldots, z_n \in \mathcal{A}$ and $\varepsilon > 0$ there is $m \in \mathbb{N}$ such that

$$\left| \varphi(z_0 \alpha_{t_1}(z_1) \ldots \alpha_{t_1+\ldots+t_n}(z_n)) - \prod_{\nu=0}^{n} \varphi(z_\nu) \right| < \varepsilon$$

provided that $\min \{ t_\nu : \nu = 1, \ldots, n \} \geq m$.

For short one may write

$$\lim_{t_1,\ldots,t_n \to +\infty} \varphi(z_0 \alpha_{t_1}(z_1) \ldots \alpha_{t_1+\ldots+t_n}(z_n)) = \prod_{\nu=0}^{n} \varphi(z_\nu) \ .$$

Proof :

Put $x_0 := \ldots := x_n := \mathbb{1}$ and $y_\nu := z_{n-\nu}$, $\nu = 1, \ldots, n$, in the definition of W^*-mixing of arbitrary degree.

Then we obtain

$$\lim_{t_1,\ldots,t_n \to -\infty} \varphi(\alpha_{t_1+\ldots+t_n}(z_0) \ldots z_n) = \prod_{\nu=0}^{n} \varphi(z_\nu) \ .$$

However, $\varphi(\alpha_{t_1+\ldots+t_n}(z_0) \ldots z_n) = \varphi(z_0 \alpha^{-1}_{t_1}(z_1) \ldots \alpha^{-1}_{t_1+\ldots+t_n}(z_n))$

yields the desired result. q.e.d

Remarks

(1) Expressions like $\varphi(z_0 \alpha_{t_1}(z_1) \ldots \alpha_{t_1+\ldots+t_n}(z_n))$ occur in the operator algebraic formulation of quantum mechanics under the name of "time-dependent Green's functions" (cf. [Br2] , p.349,398) .

(2) The mixing property derived in the preceding Lemma immediately implies that $(\mathcal{A}, \varphi, \alpha)$ is strongly mixing, i.e. for all $x, y \in \mathcal{A}$ one has

$$\lim_{|t| \to \infty} \varphi(x\, \alpha_t(y)) = \varphi(x)\,\varphi(y) \quad \text{or}$$

$$\lim_{|t| \to \infty} \alpha_t(y) = \varphi(y)\,\mathbb{1} \quad \text{with respect to } \sigma(\mathcal{A}, \mathcal{A}_*) .$$

In fact, taking $n = 1$ in the Lemma above we obtain

$$\lim_{t \to +\infty} \varphi(x\, \alpha_t(y)) = \varphi(x)\,\varphi(y) .$$

Since $\overline{\varphi(x\, \alpha_t(y))} = \overline{\varphi(\alpha_t^{-1}(x)y)} = \varphi(y^* \alpha_{-t}(x^*))$

we have the same convergence for $t \to -\infty$.

(3) It is an open problem of classical ergodic theory whether there are any measure theoretical dynamical systems which are strongly mixing, but not also mixing of arbitrary degree (cf. [Pa], p.52).

6.3.2 For showing that every W^*-K-system is W^*-mixing of arbitrary degree we proceed in two steps.

Lemma

Let $(\mathcal{A}, \varphi, \alpha; \mathcal{A}_0)$ be a W^*-K-system and define $\mathcal{A}_\infty := \bigcup \{ \mathcal{A}_t : t \geq 0 \}$.

Then for all $n \in \mathbb{N}$ and $x_0, \ldots, x_n, y_0, \ldots, y_n \in \mathcal{A}_\infty$ we have

$$\lim_{t_1, \ldots, t_n \to -\infty} \varphi(x_0\, \alpha_{t_1}(x_1) \ldots \alpha_{t_1 + \ldots + t_n}(x_n y_n) \ldots \alpha_{t_1}(y_1) y_0)$$

$$= \prod_{\nu=0}^{n} \varphi(x_\nu y_\nu) .$$

Proof:

Following the classical argument of Blum and Hanson [B1] we use the fact that $(\alpha, \varphi, \alpha; \alpha_{0]})$ satisfies property (d) of the Proposition in 6.1 :

Let $x_0, \ldots, x_n, y_1, \ldots, y_n \in \alpha_\infty \cap \alpha^1$ be given. Choose $s \geq 0$ such that $x_0, \ldots, x_n, y_0, \ldots, y_n \in \alpha_{s]}$. Given $\delta > 0$ there are $s_\nu \leq 0$ for $\nu = 1, \ldots, n$ such that

$$\sup\left\{ \left| \varphi(x_\nu z y_\nu) - \varphi(x_\nu y_\nu)\varphi(z) \right| : z \in \alpha^1_{s_\nu]} \right\} \leq \frac{\delta}{n} .$$

For $t_\nu \leq s_\nu - s \leq 0$ we have

$$\alpha_{t_\nu}(x_\nu) \ldots \alpha_{t_\nu + \ldots + t_n}(x_n y_n) \ldots \alpha_{t_\nu}(y_\nu) \in \alpha_{s_\nu]}$$

so that we can estimate

$$\left| \varphi(x_0 \ldots \alpha_{t_1+\ldots+t_n}(x_n y_n) \ldots y_0) - \prod_{\nu=0}^{n} \varphi(x_\nu y_\nu) \right|$$

$$\leq \sum_{\nu=1}^{n} \prod_{\mu=0}^{\nu-1} \left|\varphi(x_\mu y_\mu)\right| \cdot \left| \varphi(x_{\nu-1} \ldots \alpha_{t_\nu+\ldots+t_n}(x_n y_n) \ldots y_{\nu-1}) \right.$$

$$\left. - \varphi(x_{\nu-1} y_{\nu-1}) \varphi(\alpha_{t_\nu}(x_\nu) \ldots \alpha_{t_\nu+\ldots+t_n}(x_n y_n) \ldots \alpha_{t_\nu}(y_\nu)) \right|$$

$$\leq \sum_{\nu=1}^{n} \frac{\delta}{n} = \delta$$

because $\prod_{\mu=0}^{\nu-1} \left|\varphi(x_\mu y_\mu)\right| \leq 1$. q.e.d.

Remark

The fact that $\{ \alpha_t : t \in \mathbb{T} \}$ is equicontinuous with respect to the norm-topology on α makes it easy to extend the convergence from α_∞ to the norm-closure of α_∞ in α . A result to this extend can be found

in [La] . However, it requires more effort to extend the convergence to the whole of $\mathcal{A}$. To this end we prepare some results about relatively weakly compact subsets in the predual of W*-algebra.

Lemma

Let $(\mathcal{A}, \varphi, \alpha)$ be a W*-system.
For $a \in \mathcal{A}$ define the operators L_a , R_a on $\mathcal{A}_*$ by

$L_a\psi(y) := \psi(ay)$,

$R_a\psi(y) := \psi(ya)$ for all $y \in \mathcal{A}$, $\psi \in \mathcal{A}_*$.

Then for any relatively weakly compact subset $\mathcal{C}$ of $\mathcal{A}_*$ the following holds :

(1) For every $x \in \mathcal{A}$ the sets

$$\{ L_{\alpha_t(x)}\psi : t \in \mathbb{T} , \psi \in \mathcal{C} \} ,$$

$$\{ R_{\alpha_t(x)}\psi : t \in \mathbb{T} , \psi \in \mathcal{C} \}$$

are relatively weakly compact.

(2) Let $n \in \mathbb{N}$, $m \in \{1, \ldots, n\}$ and $y_1, \ldots, y_n \in \mathcal{A}$ be given. For any $\vec{s} := (s_1, \ldots, s_n, s) \in \mathbb{T}^{n+1}$ put

$a_{\vec{s}} := \alpha_{s_1}(y_1) \ldots \alpha_{s_m}(y_m)$,

$b_{\vec{s}} := \alpha_{s_{m+1}}(y_{m+1}) \ldots \alpha_{s_n}(y_n)$

and finally define $\tilde{\varphi}_{\vec{s}} \in \mathcal{A}_*$ by

$\tilde{\varphi}_{\vec{s}}(x) := \varphi(a_{\vec{s}}\, \alpha_s(x)\, b_{\vec{s}})$ for all $x \in \mathcal{A}$.

Then $\{ \tilde{\varphi}_{\vec{s}} : \vec{s} \in \mathbb{T}^{n+1} \} \subseteq \mathcal{A}_*$ is relatively weakly compact.

Proof:

First recall that $\alpha_* \mathcal{C} := \{ \alpha_{t*}(\psi) : t \in \mathbb{T}, \psi \in \mathcal{C} \}$ is relatively weakly compact (cf. [KU2] , (3.2) Proposition, p.158) . Here α_{t*}, $t \in \mathbb{T}$, denotes the preadjoint of α_t .

(1) From $\psi(\alpha_t(x)y) = \psi(\alpha_t(x\alpha_t^{-1}(y))$ it follows that

$$\{ L_{\alpha_t(x)} \psi : t \in \mathbb{T}, \psi \in \mathcal{C} \} \subseteq \alpha_* \cdot L_x \cdot \alpha_* \mathcal{C}$$

However, $\alpha_* \cdot L_x \cdot \alpha_* \mathcal{C}$ is relatively weakly compact because L_x is weakly continuous.

(2) Since $\tilde{\varphi}_{\vec{s}} = L_{a_{\vec{s}}} \cdot R_{b_{\vec{s}}} \cdot \alpha_{s*} \varphi$

a repeated application of (1) proves (2) .

q.e.d.

Now we combine the two Lemmas.

Theorem

Every W*-K-system is W*-mixing of arbitrary degree.

Proof:

For extending the convergence from α_∞ to α suppose that given $m \in \{ 0 , \ldots , n \}$ we have shown for all $\bar{x}_0 , \ldots , \bar{x}_{m-1} \in \alpha$, $x_m , \ldots , x_n \in \alpha_\infty$ and $y_0 , \ldots , y_n \in \alpha_\infty$ that

$$\varphi(\bar{x}_0 \cdots \alpha_{t_1+..+t_{m-1}}(\bar{x}_{m-1})\, \alpha_{t_1+..+t_m}(x_m) \ldots y_0)$$

converges to $\prod_{\nu=0}^{m-1} \varphi(\bar{x}_\nu y_\nu) \prod_{\mu=m}^{n} \varphi(x_\mu y_\mu)$.

Now let $\bar{x}_0, \ldots, \bar{x}_m \in \mathcal{A}^1$ and $x_{m+1}, \ldots, x_n, y_0, \ldots, y_n \in \mathcal{A}_\infty \cap \mathcal{A}^1$ be given. For each n-tupel $\vec{r} := (r_1, \ldots, r_n) \in (\mathbb{T}^-)^n$ consider the $\sigma(\mathcal{A}, \mathcal{A}_*)$-continuous functional

$$\mathcal{A} \longrightarrow \mathbb{C}\ ,\ z \longmapsto \varphi(\bar{x}_{\vec{r}}\, \alpha_{r_1+..+r_m}(z)\, x_{\vec{r}}\, y_{\vec{r}}) \quad \text{with}$$

$$\bar{x}_{\vec{r}} := \bar{x}_0 \ldots \alpha_{r_1+..+r_{m-1}}(\bar{x}_{m-1}) ,$$

$$x_{\vec{r}} := \alpha_{r_1+..+r_{m+1}}(x_{m+1}) \ldots \alpha_{r_1+..+r_n}(x_n) \quad \text{and}$$

$$y_{\vec{r}} := \alpha_{r_1+..+r_n}(y_n) \ldots y_0 .$$

According to (2) of the preceding Lemma the set $\mathcal{C} \subseteq \mathcal{A}_*$ of all such functionals with $\vec{r}$ varying over $(\mathbb{T}^-)^n$ is relatively weakly compact in $\mathcal{A}_*$. Hence so is $\mathcal{C} \cup \{R_{y_m}\varphi\}$.

Since $\mathcal{A}_\infty$ is dense in $\mathcal{A}$ with respect to the Mackey topology, for every $\varepsilon > 0$ there is $x_m \in \mathcal{A}_\infty$ such that $|\psi(\bar{x}_m - x_m)| < \varepsilon$ for all $\psi \in \mathcal{C} \cup \{R_{y_m}\varphi\}$. In particular,

$$\left| \prod_{\nu=0}^{m} \varphi(\bar{x}_\nu y_\nu) \prod_{\mu=m+1}^{n} \varphi(x_\mu y_\mu) - \prod_{\nu=0}^{m-1} \varphi(\bar{x}_\nu y_\nu) \prod_{\mu=m}^{n} \varphi(x_\mu y_\mu) \right|$$

$$\leq \left| \prod_{\nu=0}^{m-1} \varphi(\bar{x}_\nu y_\nu) \prod_{\mu=m+1}^{n} \varphi(x_\mu y_\mu) \right| \cdot \left| \varphi((\bar{x}_m - x_m) y_m) \right| < \varepsilon .$$

Finally, a 3ε -argument shows that

$\varphi(\bar{x}_{\vec{r}} \alpha_{r_1+..+r_m(\bar{x}_m)} x_{\vec{r}} y_{\vec{r}})$ converges to

$$\prod_{\nu=0}^{m} \varphi(\bar{x}_\nu y_\nu) \quad \prod_{\mu=m+1}^{n} \varphi(x_\mu y_\mu) \quad .$$

An appropriate modification of this reasoning for $y_0, \ldots, y_n \in \mathcal{A}_{\infty}$ finishes the proof.

q.e.d.

While G.G.Emch [Em1] has shown that every generalized K-flow is strongly mixing which in turn entails various weaker forms of mixing we see now that independently of the expectation hypothesis every W*-K-system enjoys an even stronger mixing property. Thus we have incorporated the concept of a W*-K-system into a hierarchy of mixing conditions analogous to the classical theory (cf. 2.1.2) .

6.3.3 In particular, every W*-K-system $(\mathcal{A}, \varphi, \alpha ; \mathcal{A}_0)$ is ergodic, i.e. Fix $(\mathcal{A}, \alpha) = \mathbb{C}\,\mathbb{1}$.
This can also be seen directly without any analysis of the mixing properties if one employs the weak* mean ergodic theorem of [Kü2] .

Proposition

Given an expanding W*-system $(\mathcal{A}, \varphi, \alpha ; \mathcal{A}_0)$ let P be the weak* mean ergodic projection of

$(\mathfrak{A}, \varphi, \alpha)$ and let ρ be the representation of $\mathbb{T}^-$ as completely positive φ-preserving contractions on $\mathfrak{A}_0$ defined by

$\rho_t := \alpha_t \restriction \mathfrak{A}_0$ for all $t \in \mathbb{T}^-$.

Then $P \restriction \mathfrak{A}_0$ is the weak* mean ergodic projection of $(\mathfrak{A}_0, \varphi, \rho)$ and

$\mathrm{Fix}(\mathfrak{A}, \alpha) = \mathrm{Fix}(\mathfrak{A}_0, \rho) \subseteq \bigcap \{ \mathfrak{A}_t : t \leq 0 \}$.

Proof :

Clearly, $\mathrm{Fix}(\mathfrak{A}, \alpha) \supseteq \mathrm{Fix}(\mathfrak{A}_0, \rho)$.

For the converse observe that by (K1) the $\sigma(\mathfrak{A}, \mathfrak{A}_*)$-closed convex hull $\overline{\mathrm{co}} \{ \rho_t(x) : t \leq 0 \}$ of the orbit of $x \in \mathfrak{A}_0$ under ρ is contained in $\mathfrak{A}_0$. Hence $P(\mathfrak{A}_0) \subseteq \mathfrak{A}_0$.

The $\sigma(\mathfrak{A}, \mathfrak{A}_*)$-continuity of P and the property $P \cdot \alpha_t = P$ for all $t \in \mathbb{T}$ imply by (K2) that $P(\mathfrak{A}) \subseteq \mathfrak{A}_0$ and so $\mathrm{Fix}(\mathfrak{A}, \alpha) = \mathrm{Fix}(\mathfrak{A}_0, \rho)$.

Since for any $x \in \mathfrak{A}_0$ we have

$\overline{\mathrm{co}} \{ \rho_t(x) : t \leq 0 \} \cap \mathrm{Fix}(\mathfrak{A}_0, \rho)$

$= \overline{\mathrm{co}} \{ \rho_t(x) : t \leq 0 \} \cap \mathrm{Fix}(\mathfrak{A}, \alpha) = \{ P(x) \}$

we see that $P \restriction \mathfrak{A}_0$ is just the weak* mean ergodic projection of $(\mathfrak{A}_0, \rho)$. q.e.d.

Appendix : The CAR-algebra, quasifree states and operators

Fix a Hilbert space $\mathfrak{K}$ with scalar product $\langle . \mid . \rangle$. Let $A(\mathfrak{K})$ be a C^*-algebra with identity $\mathbb{1}$ and $\mathfrak{K} \longrightarrow A(\mathfrak{K})$, $f \longmapsto a(f)$ an anti-linear mapping such that $A(\mathfrak{K})$ is generated by $\{ a(f) : f \in \mathfrak{K} \}$ and the canonical anti-commutation relations (CAR) hold for all $f , g \in \mathfrak{K}$:

$$a(f)\, a(g) + a(g)\, a(f) = 0 \ ,$$

$$a(f)^* a(g) + a(g)\, a(f)^* = \langle f \mid g \rangle \mathbb{1} \ .$$

Then $A(\mathfrak{K})$ is unique up to a canonical isomorphism and is called the CAR-algebra over $\mathfrak{K}$ (cf. [Br2] , p.15) .

Frequently one uses the notation $a^*(f) := a(f)^*$ for $f \in \mathfrak{K}$.

If $\mathfrak{L}$ is a closed subspace of $\mathfrak{K}$ we denote by $A(\mathfrak{L})$ the C^*-subalgebra of $A(\mathfrak{K})$ generated by $\{ a(f) : f \in \mathfrak{L} \}$.

For any $R \in \mathfrak{B}(\mathfrak{K})$, $0 \leq R \leq \mathbb{1}$, the formula

$$\omega^R(a^*(f_m) \ldots a^*(f_1) a(g_1) \ldots a(g_n)) := \delta_{mn} \det(\langle R f_\mu \mid g_\nu \rangle_{\mu\nu})$$

with $m , n \in \mathbb{N}$, $f_1 , \ldots , f_m , g_1 , \ldots , g_n \in \mathfrak{K}$ defines a state ω^R on $A(\mathfrak{K})$, called the (gauge invariant) quasifree state corresponding to R (cf. [Br2] , p.45) .

ω^R is faithful if and only if $0 \ll R \ll \mathbb{1}$, i.e.

R and $\mathbb{1} - R$ are positive definite.
For the sequel we fix $R \in \mathfrak{B}(\mathfrak{H})$, $0 \ll R \ll \mathbb{1}$.

Given a contraction $S \in \{R\}' \subseteq \mathfrak{B}(\mathfrak{H})$ there is a unique completely positive, identity-preserving operator $A^R(S)$ on $A(\mathfrak{H})$ such that for any $m, n \in \mathbb{N}$, $f_1, \ldots, f_m, g_1, \ldots, g_n \in \mathfrak{H}$ we have

$$A^R(S)\,(:a(f_m)\ldots a(f_1)a(g_1)\ldots a(g_n):_R) = :a(Sf_m)\ldots a(Sf_1)a(Sg_1)\ldots a(Sg_n):_R$$

where the symbol $: \ldots :_R$ means the normal ordering with respect to ω^R [Ev].
$A^R(S)$ is called the quasifree completely positive operator corresponding to S via R.
It leaves ω^R invariant [Ev].
If S is unitary then $A^R(S)$ is an automorphism of $A(\mathfrak{H})$ independent of R and determined simply by $A^R(S)\,a(f) = a(Sf)$ for $f \in \mathfrak{H}$ (cf. [Br2], p.16). It is called the quasifree automorphism corresponding to S and denoted by $A(S)$.

The fixed algebra of the automorphism group $\{A(z\mathbb{1}) : z \in \mathbb{C}, |z| = 1\}$ is of particular relevance in applications to physics. It is usually called the observable-algebra in $A(\mathfrak{H})$. If one wants to emphasize the difference between the observable-algebra and

$A(\mathcal{R})$ itself one sometimes calls $A(\mathcal{R})$ the field-algebra.

Let $(\pi, \mathcal{K}, \xi)$ be the GNS-representation of $A(\mathcal{R})$ with respect to ω^R. Since $A(\mathcal{R})$ is simple π is injective and so we usually identify $\pi(A(\mathcal{R}))$ and $A(\mathcal{R})$. If $\mathcal{R}$ is separable then so is $A(\mathcal{R})$ and hence $\mathcal{K}$.

The W*-algebra generated by the GNS-representation of $A(\mathcal{R})$ with respect to ω^R is defined as $\alpha^R(\mathcal{R}) := \pi(A(\mathcal{R}))'' \subseteq \mathcal{B}(\mathcal{K})$.
It is always a factor (cf. [Po], Theorem 5.1).
If $\mathcal{R}$ is separable then so is $\alpha^R(\mathcal{R})$ because $\mathcal{K}$ is separable (cf. [Sa], p.81).
We denote the normal extension of ω^R to $\alpha^R(\mathcal{R})$ by the same symbol.
For extending $A^R(S)$ from $A(\mathcal{R})$ to $\alpha^R(\mathcal{R})$ we note that $A^R(S)$ is a contraction with respect to the norm $p_{\omega R}$ (cf. Preliminaries) because $A^R(S)$ satisfies the Schwarz-inequality and leaves ω^R invariant. Hence $A^R(S)$ has a unique $p_{\omega R}$-continuous extension $\alpha^R(S)$ to the closure $\alpha^R(\mathcal{R})$ of $A(\mathcal{R})$ with respect to $p_{\omega R}$.
In particular, $\alpha^R(S)$ is a completely positive

normal operator and satisfies $\omega^R \cdot \alpha^R(S) = \omega^R$.

The modular automorphism group σ^R associated with $(\alpha^R(\mathfrak{A}) , \omega^R)$ is given as
$\sigma_t^R = \alpha^R(R^{it}(\mathbb{1} - R)^{-it})$ for all $t \in \mathbb{R}$
(cf. [Br2] , p.79).

If $P \in \mathcal{B}(\mathfrak{H})$ is a projection put
$\alpha^R(P\mathfrak{H}) := \pi(A(P\mathfrak{H}))'' \subseteq \alpha^R(\mathfrak{A})$.
It is not difficult to check that $\alpha^R(P\mathfrak{H})$ is naturally isomorphic to the W*-algebra generated by the GNS-representation of $A(P\mathfrak{H})$ with respect to the quasifree state on $A(P\mathfrak{H})$ corresponding to the operator $(PR\restriction P\mathfrak{H}) \in \mathcal{B}(P\mathfrak{H})$.
In particular, $\alpha^R(P\mathfrak{H})$ is a factor, too.

Proposition
The conditional expectation of $(\alpha^R(\mathfrak{A}) , \omega^R)$ onto $\alpha^R(P\mathfrak{H})$ exists if and only if $P R = R P$.

Proof :
By a theorem of Takesaki [Ta1] it suffices to show that σ^R leaves $\alpha^R(P\mathfrak{H})$ globally invariant if and only if $R \in \{ P \}'$.
If $R \in \{ P \}'$ then $R^{it}(\mathbb{1} - R)^{-it} \in \{ P \}'$ for all $t \in \mathbb{R}$. Hence $\sigma_t^R(A(P\mathfrak{H})) \subseteq A(P\mathfrak{H})$ and so $\sigma_t^R(\alpha^R(P\mathfrak{H})) \subseteq \alpha^R(P\mathfrak{H})$ for all $t \in \mathbb{R}$.

Conversely, if $R \notin \{P\}'$ there is $s \in \mathbb{R}$ such that $R^{is}(\mathbb{1} - R)^{-is} \notin \{P\}'$. Choose $f_1 \in P\mathfrak{H}$ such that $f_2 := R^{is}(\mathbb{1} - R)^{-is} \notin P\mathfrak{H}$ and form $f := f_2 - Pf_2$. Then $f \neq 0$ is orthogonal to $P\mathfrak{H}$ and so

$$a^*(f)\,a(f) \in \mathfrak{A}^R(P\mathfrak{H})' \cap \mathfrak{A}^R(\mathfrak{H}).$$

Suppose now $a(f_2) \in \mathfrak{A}^R(P\mathfrak{H})$. Then

$a(f) = a(f_2) - a(Pf_2) \in \mathfrak{A}^R(P\mathfrak{H})$ implies

$a^*(f)\,a(f) \in \mathfrak{A}^R(P\mathfrak{H})$ and so

$$a^*(f)\,a(f) \in \mathfrak{A}^R(P\mathfrak{H})' \cap \mathfrak{A}^R(P\mathfrak{H}).$$

Since $\mathfrak{A}^R(P\mathfrak{H})$ is a factor it follows that

$a^*(f)\,a(f) \in \mathbb{C}\mathbb{1}$, a contradiction.

Hence $\sigma_s^R(a(f_1)) = a(f_2) \notin \mathfrak{A}^R(P\mathfrak{H})$ so that $\mathfrak{A}^R(P\mathfrak{H})$ is not globally invariant under σ^R. q.e.d.

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Lebenslauf

Am 21.Februar 1953 wurde ich als erstes Kind des Lebensmittel-Einzelhändlers Werner Schröder und seiner Ehefrau Ingeborg, geborene Conrad, in Soltau, Niedersachsen geboren. Nach vierjähriger Grundschulzeit in Schneverdingen, dem Wohnort meiner Eltern, besuchte ich bis zum Abitur im Juni 1971 das naturwissenschaftlich-neusprachliche Gymnasium in Soltau.
Mein Studium begann ich im Wintersemester 1971/72 an der Universität Bremen im Studiengang "Mathematik mit naturwissenschaftlichem Anwendungsschwerpunkt". Im April 1972 wurde ich in die Studienstiftung des Deutschen Volkes aufgenommen. Nach den ersten vier Semestern ging ich für ein Jahr nach Großbritannien an die Universitäten Canterbury und Oxford. Von November 1975 bis September 1976 konnte ich an dem internationalen Projekt "Mathematische Probleme der Quantendynamik" im Zentrum für interdisziplinäre Forschung der Universität Bielefeld teilnehmen.
Die Diplom-Prüfung legte ich im Dezember 1976 an der Universität Bremen ab.
Im Mai 1977 wurde ich zum Zivildienst einberufen, den ich in einem EDV-Vorhaben am Krankenhaus Herford leistete.

Danach nahm ich mein Mathematik-Studium wieder auf und immatrikulierte mich zum Wintersemester 1978/79 mit dem Ziel der Promotion an der Universität Tübingen. Ab Juli 1979 wurde ich durch ein Promotionsstipendium der Studienstiftung des Deutschen Volkes gefördert. Im Sommersemester 1981 war ich vertretungsweise wissenschaftlicher Angestellter am Mathematischen Institut der Universität Tübingen.

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